Introduction to the Theory of Entire Functions of Several Variables

by

L. I. Ronkin

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Dedicated to my teacher
Boris Jakovlevič Levin

PREFACE

The subject of this book is the general theory of entire functions of several variables. The first work in this direction appeared as early as the beginning of this century (Borel [1], Sire [1]). However, an intensive investigation of entire functions in C^n, stemming from the general upsurge of interest in the theory of holomorphic functions of several variables, began only 25 to 30 years later. An important part in this respect was played by Lelong's work, particularly his studies of plurisubharmonic functions. The methods of the theory of plurisubharmonic functions provided a general approach to several problems in the theory of entire functions and led to substantial results. Our first two chapters therefore present the general theory of subharmonic and plurisubharmonic functions and a few related specific subjects. In Chapters 3 and 4, devoted to the theory of entire functions proper, particular attention is paid to questions of growth and distribution of zeros of entire functions of several variables.

To make the book self-dependent, we have added an introduction presenting certain material from the theory of functions of a real variable, the theory of distributions, etc., not usually included in general university courses. The same purpose is served by § 4 of Chapter 1 which contains material from potential theory, and by § 1 of Chapter 4, which presents a few results from the theory of analytic sets.

The material assembled in this book presents highly varying degrees of difficulty. In this connection, we might mention that an acquaintance with §§ 4 and 5 of Chapter 1, §§ 3 and 7 of Chapter 2, and § 2 of Chapter 3 is not necessary for an understanding of the other parts of the book. The same applies to § 6 of Chapter 3.

I am indebted to all participants in B. Ja. Levin's seminar, in which the results presented in this book have often been discussed, for friendly support and useful discussion. Thanks are also due to B. V. Šabat, who read the manuscript and offered much valuable advice, and to V. P. Zaharjuta, whose remarks were used in § 6.4 of Chapter 3.

I am particularly grateful to V. È. Kacnel'son and N. S. Landkof, who read the book attentively in manuscript. Thanks to their remarks, several inaccuracies in the original text have been eliminated.

§ 6 of Chapter 3 was included in the book on the advice of V. P. Palamodov.

* *Translation editor's note.* This translation incorporates corrections furnished by the author.

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INTRODUCTION

§ 1. Standard notation

We denote complex and real n-space by C^n and R^n, respectively. Thus, $z \in C^n$ means that $z = (z_1, \cdots, z_n)$, where $z_1, \cdots, z_n$ are complex numbers. Similarly, $t \in R^n$ means that $t = (t_1, \cdots, t_n)$, where $t_1, \cdots, t_n$ are real numbers.

As a rule, unless explicitly stated otherwise, we shall denote $\operatorname{Re} z_j = x_j$ and $\operatorname{Im} z_j = y_j$, so that $z = x + iy$, where $x \in R^n$, $y \in R^n$. To emphasize that points in C^n have been denoted by z, ζ, etc., we shall sometimes write $C^n_{(z)}$, $C^n_{(\zeta)}$, etc., instead of C^n. Similarly, R^n will be replaced by $R^n_{(x)}$, $R^n_{(y)}$, etc.

The length of a real or complex vector $A = (A_1, \cdots, A_n)$ will be denoted by $|A|$. Thus

$$|A| = \sqrt{|A_1|^2 + \cdots + |A_n|^2}.$$

If the coordinates A are nonnegative, we set

$$\|A\| = A_1 + \cdots + A_n,$$

and if they are also integers, then

$$A! = A_1! \cdots A_n!.$$

If A and B are two vectors of the same dimensionality, we set

$$\langle A, B \rangle = A_1 B_1 + \cdots + A_n B_n$$

and

$$A/B = (A_1/B_1, \cdots, A_n/B_n).$$

Here the coordinates of each of the vectors A and B may be either real or complex. If the coordinates of B are nonnegative, we shall also use the notation

$$A^B = A_1^{B_1} \cdots A_n^{B_n}.$$

The letter k will always denote a vector with nonnegative integer coordinates.

The set of all points P belonging to a set M and satisfying a condition C will be denoted by

$$\{P : P \in M, C\}.$$

Thus the positive hyperoctant R^n_+ in R^n will be

$$R^n_+ = \{x : x \in R^n, x_1 \geqq 0, \cdots, x_n \geqq 0\}.$$

If M is a set in some topological space, then, as usual, $\bar{M}$ will denote the closure of M and ∂M its boundary. The notation $A \Subset M$ will mean that A is relatively compact in M, i.e. $\bar{A}$ is compact and $\bar{A} \subset M$. $A \backslash B$ will denote the complement of the set $B \cap A$ in A, and $\varnothing$, the empty set. We shall also use the universal quantifier $\forall$ to stand for the words "for any···".

If a mapping α defined on a set M maps points $x \in M$ onto points $y = \alpha x$ in a set N, we write $\alpha : M \to N$.

The image of any subset M_1 of M will be denoted by αM_1, and the preimage of a subset $N_1 \subset N$, i.e. the set $\{x : x \in M, \alpha x \in N_1\}$, by $\alpha^{-1} N_1$.

We shall continue to introduce new notation as the need arises in the text. Notation defined in this Introduction will have the same meaning throughout the book. Similarly, notation introduced at the beginning of some part of the book (section or chapter) will retain its sense throughout that part, unless otherwise stated.

§ 2. Semicontinuous functions[1)]

A function $f(x)$, defined on some set $M \subset R^n$ and assuming values in $[-\infty, \infty)$, is said to be *upper semicontinuous at a point* $x_0 \in M$ if for any number $A > f(x_0)$ there exists a number $\delta = \delta(x_0, A)$ such that $f(x) < A$ whenever $|x - x_0| < \delta$ and $x \in M$. A function $f(x)$ is said to be *upper semicontinuous on the set* M if it is upper semicontinuous at each point of M.

The following propositions follow immediately from the definition.

(a) *The sum of a finite number of upper semicontinuous functions is upper semicontinuous.*

(b) *The product of an upper semicontinuous function and a positive constant is again upper semicontinuous.*

(c) *If the functions* $f_1(x), \cdots, f_m(x)$, $m < \infty$, *are upper semicontinuous, then* $f(x) = \max_{1 \leq i \leq m} f_i(x)$ *is also upper semicontinuous.*

(d) *If a function* $f(x)$ *is upper semicontinuous at a point* x_0, *then*

$$\overline{\lim_{x \to x_0}} f(x) \leqq f(x_0).$$

(e) *If the function* $f(x)$ *is upper semicontinuous on a compact set* K, *then* $\sup_K f(x) < \infty$, *and there exists a point* $x_0 \in K$ *such that* $f(x_0) = \sup_K f(x)$.

(f) *The limit of a uniformly convergent sequence of upper semicontinuous functions is an upper semicontinuous function.*

The following test for semicontinuity will be used frequently in the sequel.

[1)] For more details on semicontinuous functions, see, for example, Natanson [1].

Let $f(x)$ be a function defined on a set $M \subset R^n$ with values in $[-\infty, \infty)$, such that

$$f(x) = \lim_{m\to\infty} f_m(x),$$

where the functions $f_m(x)$ are continuous on M and form a monotone decreasing sequence. Then $f(x)$ is upper semicontinuous on M.

Indeed, for any x_0 and any $A > f(x_0)$ there obviously exists a number $m_0 = m_0(x_0, A)$ such that $f_m(x_0) < A$ for all $m \geqq m_0$. Since $f_{m_0}(x)$ is continuous, there exists $\delta = \delta(A, m_0, x_0) > 0$ such that $f_{m_0}(x) < A$ for $|x - x_0| < \delta$, $x \in M$. But the sequence $\{f_m(x)\}$ is monotone decreasing, and hence $f(x) < A$ for $|x - x_0| < \delta$, $x \in M$, so that $f(x)$ is upper semicontinuous at the point x_0.

The converse is also valid:

If $f(x)$ is upper semicontinuous on a compact set $K \subset R^n$, there exists a monotone decreasing sequence of functions, continuous on K, which converges to $f(x)$.

We first prove this assertion under the additional assumption $f(x) \not\equiv -\infty$ for all $x \in K$. Set

$$f_m(x) = \max_{y\in K} \{f(y) - m|x - y|\},$$

and let $y_m = y_m(x)$ be a point y such that

$$f_m(x) = f(y_m) - m|x - y_m|. \tag{0.2.1}$$

We claim that each function $f_m(x)$ is continuous on K. Indeed, by definition of the function $f_m(x)$ and the point $y_m(x)$, for any $x' \in K$ we have

$$\begin{aligned} f_m(x) &\geqq f(y_m(x')) - m|x - y_m(x')| \\ &\geqq f(y_m(x')) - m|y_m(x') - x'| - m|x - x'| = f_m(x') - m|x - x'|. \end{aligned}$$

Similarly,

$$f_m(x') \geqq f_m(x) - m|x - x'|.$$

Consequently

$$|f_m(x) - f_m(x')| \leqq m|x - x'|$$

and thus the function $f_m(x)$ is continuous on K.

Now note that the sequence of functions $f_m(x)$ is monotone decreasing and hence the limit $g(x) = \lim_{m\to\infty} f_m(x)$ exists. Since

$$f_m(x) \geqq f_m(x') - m|x - x'|$$

for any $x' \in K$, it follows that $f_m(x) \geqq f(x)$, and so

$$g(x) \geqq f(x). \tag{0.2.2}$$

Consequently $g(x) > -\infty$. Hence, letting $m \to \infty$ in (0.2.1) and recalling that the function $f(x)$ is bounded, we see that $\lim_{m\to\infty} y_m(x) = x$. Therefore

$$\lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} \{f(y_m) - m|y_m - x|\} \leqq \overline{\lim_{m\to\infty}} f(y_m) \leqq f(x);$$

that is, $g(x) \leqq f(x)$. In view of (0.2.2), this implies that $g(x) = f(x)$, and thus

$$f(x) = \lim_{m\to\infty} f_m(x).$$

The assertion is thus proved provided that $f(x) > -\infty$ for all $x \in K$. To do away with this restriction, note that if $f(x)$ is any upper semicontinuous function, then $e^{f(x)}$ is a nonnegative upper semicontinuous function. Thus there exists a monotone decreasing sequence of continuous functions $\varphi_m(x)$ such that

$$e^{f(x)} = \lim_{m\to\infty} \varphi_m(x).$$

But then

$$f(x) = \lim_{m\to\infty} \ln\left(\frac{1}{m} + \varphi_m(x)\right).$$

The functions $\ln(1/m + \varphi_m(x))$ are clearly continuous and form a monotone decreasing sequence.

§ 3. Convex sets and convex functions

A set $B \subset R^n$ is said to be *convex* if, together with any two points $x, y \in B$, it also contains the entire segment joining these points, i.e. the points $t = \lambda x + \mu y$, where $\lambda \in [0, 1]$, $\lambda + \mu = 1$.

A function $f(x)$, $-\infty < f(x) < \infty$, defined on a convex set $B \subset R^n$ is said to be *convex* if, for any $\lambda \in [0, 1]$, $\mu = 1 - \lambda$ and any $x, y \in B$ we have

$$f(\lambda x + \mu y) \leqq \lambda f(x) + \mu f(y).$$

The following properties of convex functions are important.

(a) *A convex function is continuous at any interior point of its domain of definition.*

Indeed, let x_0 be an interior point of a convex set B on which $f(x)$ is defined. Then there exists an n-simplex with vertices $x^{(1)}, \cdots, x^{(n+1)}$ [2] which contains x_0 as an interior point. Since $f(x)$ is a convex function, the following inequality holds at each point of this simplex:

$$f(\lambda_1 x^{(1)} + \cdots + \lambda_{n+1} x^{(n+1)}) \leqq \sum_{j=1}^{n+1} \lambda_j f(x^{(j)}) \leqq \max_{1\leqq j\leqq n+1} f(x^{(j)}).$$

Consequently the function $f(x)$ is bounded above on the simplex, and hence also in a suitable neighborhood of the point x_0. We claim that the function is also bounded below in a neighborhood of x_0. Indeed, otherwise there exists a

[2] Recall that an n-simplex is defined in terms of its vertices $x^{(1)}, \cdots, x^{(n+1)}$ as the set of all points $x = \lambda_1 x^{(1)} + \cdots + \lambda_{n+1} x^{(n+1)}$ where $\lambda_1, \cdots, \lambda_{n+1} \geqq 0$ and $\lambda_1 + \cdots + \lambda_{n+1} = 1$.

sequence of points $y_j \to x_0$, $j \to \infty$, such that

$$\lim_{j\to\infty} f(y_j) = -\infty.$$

Consider the simplex spanned by the points $x^{(1)}, \cdots, x^{(n)}$ and y_j. At each of its points, i.e. at each point

$$x = \lambda_1 x^{(1)} + \cdots + \lambda_n x^{(n)} + \lambda_{n+1} y_j, \tag{0.3.1}$$

where $\lambda_1 \geqq 0, \cdots, \lambda_{n+1} \geqq 0$ and $\lambda_1 + \cdots + \lambda_{n+1} = 1$, it follows from the convexity of $f(x)$ that

$$f(x) \leqq \lambda_1 f(x^{(1)}) + \cdots + \lambda_n f(x^{(n)}) + \lambda_{n+1} f(y_j). \tag{0.3.2}$$

The numbers $\lambda_1, \cdots, \lambda_{n+1}$ defining the point x are continuous functions of the position of the vertices $x^{(1)}, \cdots, x^{(n)}$ and y_j of the simplex. Thus for each interior point x of the limit simplex with vertices $x^{(1)}, \cdots, x^{(n)}$ and x_0 the numbers λ_{n+1} in (0.3.1) converge to a nonzero number as $j \to \infty$. Hence by (0.3.2) it follows that $f(x) = -\infty$. This is impossible, since $x \in B$. Thus the function $f(x)$ is bounded in some neighborhood of x_0.

Set $|f(x_0 + h)| \leqq M < \infty$ for $|h| \leqq r$. Then for $\lambda, \mu > 0$, $\lambda + \mu = 1$, we have

$$f(x_0 + \mu h) \leqq \lambda f(x_0) + \mu f(h + x_0).$$

Hence it follows that

$$f(x_0 + \mu h) - f(x_0) \leqq \mu[f(x_0 + h) - f(x_0)] \leqq 2\mu M.$$

Similarly, $f(x_0) - f(x_0 + \mu h) \leqq 2\mu M$. Consequently

$$|f(x_0 + \mu h) - f(x_0)| \leqq 2\mu M,$$

whence, setting $|h| = r$ and $\mu h = \Delta x$, we get

$$|f(x_0 + \Delta x) - f(x_0)| \leqq 2|\Delta x| M/r.$$

The continuity of $f(x)$ at x_0 is now obvious.

In view of their simplicity, the following properties (b), (c) and (d) are stated without proof.

(b) *A twice continuously differentiable function $f(x)$ is convex in a domain $B \subset R^n$ if and only if the following inequality holds in B:*

$$\inf_{R_{(\xi)}^n} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \xi_i \xi_j \geqq 0.$$

(c) *Let the functions $f_j(x)$, $j = 1,2, \cdots$, be convex on a set $B \subset R^n$ and have a limit $f(x) = \lim_{j\to\infty} f_j(x)$. Then $f(x)$ is also convex on B.*

(d) *Let the function $f(x,y)$ be defined on the cartesian product of a convex set $B \subset R^n$ and a domain $G \subset R^m$. Assume that for any fixed y the function $f(x,y)$ is convex on B, and for any fixed $x \in B$ the integral*

$$\varphi(x) = \int_G f(x, y)d\mu_y,$$

where μ is a measure on G, exists. Then $\varphi(x)$ is convex on B.

Convex sets are sometimes conveniently characterized in terms of "support functions."

DEFINITION. The *support function* of a convex set $B \subset R^n$ is the function

$$H_B(\lambda) = \sup_{x \in B} (\lambda_1 x_1 + \cdots + \lambda_n x_n) \qquad (\lambda \in R^n).$$

For $|\lambda| = 1$ this function has a simple geometrical interpretation: the sum $\langle \lambda, x \rangle = \lambda_1 x_1 + \cdots + \lambda_n x_n$ is simply the projection of the vector x on the direction of the vector λ, and therefore $H_B(\lambda)$ is the distance from the origin to the nearest hyperplane perpendicular to λ such that the half-space $\{x: \langle \lambda, x \rangle > H_B(\lambda)\}$, whose boundary is the hyperplane in question, contains no points of B. Such hyperplanes are called *support hyperplanes.*

It can be shown that through each point $x \in \partial B$ there is at least one support hyperplane; and, conversely, if B is bounded, each support hyperplane contains points of ∂B.

If a convex set B is bounded, the function $H_B(\lambda)$ is obviously bounded on any finite set in the space $R^n_{(\lambda)}$. But if B is not bounded, the function $H_B(\lambda)$ becomes infinite for certain values of λ. Let D_B denote the set of λ such that $H_B(\lambda) < \infty$. The dimensionality of this set may be lower than that of the space $R^n_{(\lambda)}$, i.e. less than n. We claim that the set D_B is convex and the function $H_B(\lambda)$ is convex on D_B. Indeed, it follows from the definition of a support function that for any $\lambda^{(1)}$ and $\lambda^{(2)}$ and any $t > 0$ it satisfies the following conditions: [3]

(1) $H_B(t\lambda) = tH_B(\lambda)$ (*homogeneity*),

(2) $H_B(\lambda^{(1)} + \lambda^{(2)}) \leqq H_B(\lambda^{(1)}) + H_B(\lambda^{(2)})$ (*subadditivity*),

and hence also the condition

$$H_B(t_1\lambda^{(1)} + t_2\lambda^{(2)}) \leqq t_1 H_B(\lambda^{(1)}) + t_2 H_B(\lambda^{(2)}) \tag{0.3.3}$$

for any $t_1, t_2 \geqq 0$. (The function $H_B(\lambda)$ is allowed to have infinite values.) It follows from (0.3.3) that the set D_B and function $H_B(\lambda)$ are convex.

Now let L be the smallest linear subspace of R^n containing the set D_B. Call a point $\lambda \in D_B$ an interior point of D_B if it is an interior point of this set as a subset of L. It follows from the above properties of convex functions that the function $H_B(\lambda)$ is continuous at each interior point of D_B. In particular, the support function of a bounded domain is everywhere continuous in R^n. Note the following additional properties of support functions:

---

[3] For $n = 2$ these conditions are equivalent to the condition that the function be expressible as $r\varphi(\theta)$ (r, θ are polar coordinates in R^2) where $\varphi(\theta)$ is a trigonometrically convex function, i.e.

$$\varphi(\theta_2) \leqq \frac{\varphi(\theta_1)\sin(\theta_3 - \theta_2) + \varphi(\theta_3)\sin(\theta_2 - \theta_1)}{\sin(\theta_3 - \theta_1)}$$

for all $\theta_1 \leqq \theta_2 \leqq \theta_3$, $\theta_3 - \theta_2 < \pi$, $\theta_2 - \theta_1 < \pi$.

(a) *If* $B \subset G$, *then* $H_B(\lambda) \leqq H_G(\lambda)$.

(b) *If* B *is the smallest convex domain containing the convex domains* $B_1, \cdots, B_m$ (*the convex hull of the domains* $B_1, \cdots, B_m$), *then*

$$H_B(\lambda) = \max_{1 \leqq i \leqq m} H_{B_i}(\lambda).$$

(c) $H_B(\lambda) + H_B(-\lambda) \geqq 0 \ \forall \lambda \in R^n$.

(d) *Any closed convex set* $B \subset R^n$ *is expressible as*

$$B = \bigcap_{\lambda \in D_B} \{x: \langle \lambda, x \rangle \leqq H_B(\lambda)\}.$$

Moreover, as is easily seen, *the points* λ *in this representation may be restricted to any countable set of interior points of* D_B *which is dense in the intersection of* D_B *and the unit sphere.*

The following property, though quite simple, is less obvious:

(e) *Let* K_1 *and* K_2 *be convex compact sets whose union* $K_1 \cup K_2$ *is convex. Then the support function of their intersection* K *satisfies the condition*

$$H_K(\lambda) = \min\{H_{K_1}(\lambda), H_{K_2}(\lambda)\}. \tag{0.3.4}$$

Indeed, suppose that for some λ^0

$$H_K(\lambda^0) < \min\{H_{K_1}(\lambda^0), H_{K_2}(\lambda^0)\}. \tag{0.3.5}$$

Then the half-space $\{x: \langle \lambda^0, x \rangle > H_K(\lambda^0)\}$ contains points $x' \in K_1$ and $x'' \in K_2$. Since the union $K_1 \cup K_2$ is convex, it contains the segment joining these points. Hence, since the sets K_1 and K_2 are closed, it follows that the segment joining x' and x'' contains points of the compact set K. This contradicts the definition of support function, and shows that inequality (0.3.5) cannot hold. The inequality

$$H_K(\lambda) \leqq \min\{H_{K_1}(\lambda), H_{K_2}(\lambda)\}$$

is obvious, and thus (0.3.4) must hold.

To end this section, we prove that *any function* $H(\lambda)$ *taking finite values in* R^n *and satisfying conditions* (1) *and* (2) (*homogeneity and subadditivity*) *is the support function of some bounded convex set.* Indeed, we have

$$H(\lambda) = H_B(\lambda), \tag{0.3.6}$$

where

$$B = \bigcap_{|\lambda|=1} \{x: \langle \lambda, x \rangle \leqq H(\lambda)\}.$$

We shall prove this assertion[4] for $n = 2$, since this is the simplest case and is that needed in the sequel.

We first observe that $H(\lambda) = H(\lambda_1, \lambda_2)$, as a convex function, is continuous

[4] The proof presented here is due to Levin [1].

throughout R^2. Equality (0.3.6) will obviously be proved if we can show that every straight line

$$\lambda_1 x_1 + \lambda_2 x_2 - H(\lambda_1, \lambda_2) = 0 \tag{0.3.7}$$

contains at least one point of the set B. Denote this straight line by $l(\lambda)$, fix $\lambda = \lambda^0$, and consider in the plane $R^2_{(\lambda)}$ the two open half-planes P_1 and P_2 whose common boundary is the straight line $\{\lambda: \lambda = t\lambda^0, -\infty < t < \infty\}$. Denote

$$A_1 = l(\lambda^0) \cap \Big\{ \bigcup_{\lambda' \in P_1} \{x: \lambda_1' x_1 + \lambda_2' x_2 > H(\lambda')\} \Big\},$$

$$A_2 = l(\lambda^0) \cap \Big\{ \bigcup_{\lambda'' \in P_2} \{x: \lambda_1'' x_1 + \lambda_2'' x_2 > H(\lambda'')\} \Big\}.$$

We claim that $A_1 \cap A_2 = \varnothing$. Indeed, if $(x_1, x_2) \in A_1$ and $(x_1, x_2) \in A_2$, then for some $\lambda' \in P_1$ and $\lambda'' \in P_2$ we have

$$\lambda_1' x_1 + \lambda_2' x_2 > H(\lambda'), \qquad \lambda_1'' x_1 + \lambda_2'' x_2 > H(\lambda''). \tag{0.3.8}$$

Note that $\lambda' \neq -\lambda''$, since otherwise it would follow from (0.3.8) that

$$0 > H(\lambda') + H(\lambda'') \geqq H(\lambda' + \lambda'') = 0.$$

Therefore, as is readily seen, the vectors λ' and λ'' are linearly independent and so, for some t and s,

$$\lambda^0 = t\lambda' + s\lambda''.$$

Since $\lambda' \in P_1$ and $\lambda'' \in P_2$, it follows that the numbers t and s have the same sign. Suppose that $t < 0$ and $s < 0$. Then, using inequalities (0.3.8) and the properties of the function $H(\lambda)$, we get

$$\begin{aligned}(|t|\lambda_1' + |s|\lambda_1'')x_1 + (|t|\lambda_2' + |s|\lambda_2'')x_2 \\ > H(|t|\lambda' + |s|\lambda'') \geqq -H(t\lambda' + s\lambda'') = -H(\lambda^0).\end{aligned}$$

Hence it follows that $\lambda_1^0 x_1 + \lambda_2^0 x_2 < H(\lambda^0)$, contradicting the fact that (x_1, x_2) lies on the straight line $l(\lambda^0)$.

Now suppose that $t > 0$ and $s > 0$. It then follows from (0.3.8) that

$$\begin{aligned}(t\lambda_1' + s\lambda_1'')x_1 + (t\lambda_2' + s\lambda_2'')x_2 \\ < tH(\lambda') + sH(\lambda'') \geqq H(t\lambda' + s\lambda'') = H(\lambda^0)\end{aligned}$$

and hence $\lambda_1^0 x_1 + \lambda_2^0 x_2 > H(\lambda^0)$, which also contradicts the choice of the point (x_1, x_2) on the straight line $l(\lambda^0)$. Consequently $A_1 \cap A_2 = \varnothing$. Each of the sets A_1 and A_2 is an open ray, and therefore the straight line $l(\lambda^0)$ contains at least one point not in either of these sets. This point clearly belongs to all half-planes $\{x: \lambda_1 x_1 + \lambda_2 x_2 \leqq H(\lambda)\}$, and is therefore a point of the set B. Consequently $H(\lambda^0) = H_B(\lambda^0)$. This completes the proof of (0.3.6).

§ 4. On the growth of functions

In investigation of entire functions, whether of one variable or more, one must often characterize the growth of certain auxiliary functions derived from the original entire function. The simplest of these are functions of a single real variable, which are often monotone nondecreasing. An example is the function $n_f(t)$, defined as the number of zeros of an entire function $f(z)$, $z \in C^1$, in the disk $|z| \leqq t$. Before considering entire functions, therefore, it is natural to examine arbitrary functions on a half-axis.

Let $f(t)$ be a nonnegative function on $(0, \infty)$. In order to characterize its rate of growth, we compare it with the functions μt^λ.

The greatest lower bound of the set of numbers $\lambda \geqq 0$ such that $f(t) < t^\lambda$ for $t \to +\infty$[5] is known as the *order* of the function $f(t)$. The order of the function $f(t)$ is usually denoted by $\rho(f)$ or simply ρ.

It is clear that the order of a function can be computed from the formula

$$\rho(f) = \overline{\lim_{t\to\infty}} \frac{\ln f(t)}{\ln t}.$$

The type of a function $f(t)$ for order $\rho > 0$ is defined as the greatest lower bound $\sigma(f, \rho)$ of the set of numbers $\mu \leqq \infty$ such that $f(t) < \mu t^\rho$ for $t \to +\infty$. We shall often replace $\sigma(f, \rho)$ by the briefer notation $\sigma(f)$ or simply σ. It is easy to see that

$$\sigma(f, \rho) = \overline{\lim_{t\to\infty}} \frac{f(t)}{t^\rho}.$$

Functions $f(t)$ such that $\sigma(f) = 0$, $0 < \sigma(f) < \infty$, and $\sigma(f) = \infty$ are known respectively as functions of *minimal*, *normal*, and *maximal* type.

Another characterization of the growth of a function $f(t)$, apart from its order and type, is the behavior (convergence or divergence) of the integral

$$\int^\infty \frac{f(t)}{t^{\rho+1}}\, dt. \tag{0.4.1}$$

Note that if we replace the order ρ in this integral by an arbitrary number $\alpha > \rho(f)$ the result is obviously a convergent integral. However, if $f(t)$ is monotone nondecreasing, $\int^\infty (f(t)/t^{\alpha+1})\, dt$ will diverge if $\alpha < \rho(f)$ or $\alpha = \rho(f)$ and $\sigma(f, \rho) > 0$. Indeed, in this case there exists a sequence of numbers t_j such that for any j we have $t_{j+1} > 2t_j$ and for some $a > 0$

$$f(t_j) \geqq a t_j^0, \qquad j = 1, 2, \cdots.$$

Since the function $f(t)$ is monotone,

[5] An inequality $F(t) < 0$ is said to hold for $t \to t_0$ if it holds in some neighborhood of t_0.

$$\int_{t_1}^{\infty} \frac{f(t)}{t^{\rho+1}}\,dt = \sum_{j=1}^{\infty} \int_{t_j}^{t_{j+1}} \frac{f(t)}{t^{\rho+1}}\,dt \geqq \frac{a}{\rho} \sum_{j=1}^{\infty} t_j^{\rho}\left(\frac{1}{t_j^{\rho}} - \frac{1}{t_{j+1}^{\rho}}\right) \geqq \frac{a}{\rho} \sum_{j=1}^{\infty} \left(1 - \left(\frac{1}{2}\right)^{\rho}\right) = \infty.$$

The assertion just proved may be formulated as follows:

If a nonnegative monotone nondecreasing function $f(t)$ satisfies the condition

$$\int^{\infty} \frac{f(t)}{t^{\alpha+1}}\,dt < \infty,$$

then

$$\lim_{t\to\infty} \frac{f(t)}{t^{\alpha}} = 0.$$

Thus characterization of the growth of functions in terms of the integral (0.4.1) is meaningful only for functions of minimal type.

Let us say that nonnegative functions $f(t)$ and $\varphi(t)$ belong to the same *convergence class* if the integrals

$$\int^{\infty} \frac{\varphi(t)}{t^{\alpha+1}}\,dt, \qquad \int^{\infty} \frac{f(t)}{t^{\alpha+1}}\,dt$$

converge (and therefore also diverge) for exactly the same values of α. We shall also say that a function $\varphi(t)$ *increases faster* than $f(t)$, if the order of $\varphi(t)$ is greater than that of $f(t)$, or their orders are equal and $\varphi(t)$ is of maximal (normal) type and $f(t)$ of normal (minimal) type, or if both functions are of the same order and minimal type but

$$\int^{\infty} \frac{\varphi(t)}{t^{\rho+1}}\,dt = \infty \qquad \text{and} \qquad \int^{\infty} \frac{f(t)}{t^{\rho+1}}\,dt < \infty.$$

Similarly one defines the sense of the expressions the "function $\varphi(t)$ *increases no faster* (or *no slower*) than the function $f(t)$" and "the function $\varphi(t)$ has *the same rate of growth* as the function $f(t)$."

In many cases (as we shall see later), it is convenient to replace $f(t)$ by the function

$$F(t) = \int_0^t \frac{f(s)}{s}\,ds,$$

which is a kind of average of $f(t)$. The function $f(t)$ is assumed here to be monotone nondecreasing and such that for some $t_0 > 0$ we have $f(t) = 0$ for all $t \in (0, t_0)$. The essential point here is that *when $\rho(f) > 0$ the functions $f(t)$ and $F(t)$ have the same rate of growth.* Indeed, that they have the same order and type follows from the obvious estimates $f(t) \leqq F(et)$ and

$$F(t) \leqq \lambda\mu t^{\lambda}, \qquad \text{if } f(t) \leqq \mu t^{\lambda} \qquad (\lambda > 0),$$

and the fact that when $\sigma(f) = \sigma(F) = 0$ the functions $f(t)$ and $F(t)$ belong to the same convergence class follows from the equality

$$\int_{}^{\infty} \frac{F(t)}{t^{\rho+1}}\,dt = -\frac{1}{\rho}\,\frac{F(t)}{t^{\rho}}\Big|^{\infty} + \frac{1}{\rho}\int_{}^{\infty} \frac{f(t)}{t^{\rho+1}}\,dt.$$

Another function often encountered in the theory of entire functions, defined, as is $F(t)$, in terms of the original function $f(t)$, is

$$K_q(t;f) = t^q\left\{\int_0^t \frac{f(s)}{s^{q+1}}\,ds + t\int_t^{\infty} \frac{f(s)}{s^{q+2}}\,ds\right\},$$

where q is the smallest integer such that the second integral converges. As before, one assumes here that $f(t)$ is monotone nondecreasing and $f(t) = 0$ for all $t \in (0, t_0)$, $t_0 > 0$.

We claim that the order of the function $K_q(t; f)$ cannot exceed that of $f(t)$, and when $\rho(f) \neq q$ the type of the function $K_q(t; f)$ for order $\rho(f)$ (minimal, normal or maximal) does not exceed the corresponding type of $f(t)$.[6] To prove this, suppose that for some $\lambda \in (q, q+1)$, some positive $\mu < \infty$, and all sufficiently large t we have $f(t) \leqq \mu t^{\lambda}$. Then $K_q(t; f)$ satisfies the estimate

$$\begin{aligned} K_q(t;f) &\leqq C_1 + \mu t^q\left(\int_{t_0}^t s^{\lambda-q-1}\,ds + t\int_t^{\infty} s^{\lambda-q-2}\,ds\right) \\ &= C_2 + \mu\left(\frac{1}{\lambda-q} + \frac{1}{q+1-\lambda}\right)t^{\lambda}. \end{aligned}$$

Here C_1 and C_2 are constants. These estimates clearly imply our assertion for $\rho(f) < q+1$. If $\rho = q+1$, then, since the integral (0.4.1) is convergent in this case, we have

$$\lim_{t\to\infty} \int_t^{\infty} \frac{f(s)}{s^{q+2}}\,ds = 0, \qquad \lim_{s\to\infty} \frac{f(t)}{t^{q+1}} = 0,$$

and, as follows from the last equality,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{f(s)}{s^{q+1}}\,ds = 0.$$

In this case, therefore, the function $K_q(t; f)$ is of at most minimal type for order ρ. This completes the proof of our assertion concerning the relation between the rates of growth of the functions $f(t)$ and $K_q(t; f)$.

REMARK. If the function $f(t)$ is not monotone nondecreasing, but can be expressed as $f(t) = \gamma(t)t^{-\alpha}$, where $\gamma(t)$ is a monotone nondecreasing function and $\alpha > 0$, it is readily seen that the above assertions on the growth of the functions $F(t)$ and $K_q(t; f)$ remain valid. It is also obvious that $f(t)$ is again of minimal type if the integral (0.4.1) is convergent.

[6] In other words, $\sigma(K_q, \rho) \leqq C\sigma(f, \rho)$, where C is a positive constant depending on $f(t)$.

§ 5. Distributions

We present here some necessary material from the theory of distributions. A detailed exposition may be found, e.g., in Schwartz [1], Gel'fand and Šilov [1], or Landkof [1].

1. *Spaces of test functions and distributions.* Let G be a domain in the space R^n. A function $f(x)$ defined in G is said to have *compact support in* G if $f(x) \equiv 0$ off some domain $G_1 \Subset G$. Let $D(G)$ denote the space of all real[7] infinitely differentiable functions with compact support in G. The elements of the space $D(G)$ are called *test functions,* and $D(G)$ is accordingly called the *space of test functions* or *test space.* If $G_1 \subset G_2$, we always assume that $D(G_1) \subset D(G_2)$, defining the functions $\varphi \in D(G_1)$ to be zero in $G_2 \backslash G_1$. The closure of the set of points x at which a test function $\varphi(x)$ does not vanish is known as its *support* and denoted by supp φ. Convergence in the space $D(G)$ is defined as follows.

A sequence of test functions $\{\varphi_j(x)\}$ is said to *converge in* $D(G)$ to a function $\varphi(x) \in D(G)$ if the supports of all functions $\varphi_j(x)$ are subsets of the same domain $G_1 \Subset G$ and the functions $\varphi_j(x) - \varphi(x)$ converge uniformly to zero together with their derivatives of all orders.

Now let $f(x)$ be an arbitrary real function which is integrable in every domain $G_1 \Subset G$. For any function $\varphi \in D(G)$ the integral

$$(f, \varphi) = \int_G f(x)\varphi(x)dx \tag{0.5.1}$$

exists. Thus the function $f(x)$ defines a functional on the space $D(G)$.[8] It is obvious that for any real α_1 and α_2 and any test functions $\varphi_1(x)$ and $\varphi_2(x)$ we have

$$(f, \alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1(f, \varphi_1) + \alpha_2(f, \varphi_2).$$

Moreover, it is also obvious that if test functions $\varphi_j(x)$ converge in $D(G)$ to a function $\varphi(x) \in D(G)$, then

$$\lim_{j\to\infty} (f, \varphi_j) = (f, \varphi).$$

This means that the functional defined in terms of $f(x)$ by (0.5.1) is linear and continuous on $D(G)$. However, not all continuous linear functionals on $D(G)$ can be obtained in this way. In particular, functionals defined on $D(G)$ by an equality

$$(\mu, \varphi) = \int_G \varphi(x)d\mu(x), \tag{0.5.2}$$

where μ is a certain measure (mass distribution) in the domain G, are generally not of this type. Recall that a *measure in the domain G is a nonnegative countably additive set function, defined on all Borel subsets of G and finite on all Borel subsets*

[7] This restriction is not essential.

[8] Note that the set of values of this functional on all functions $\varphi \in D(G)$ determines the function $f(x)$ uniquely, up to a function which vanishes almost everywhere.

which are relatively compact in G. It is easy to see that any functional defined in terms of a measure μ by (0.5.2) is linear and continuous on $D(G)$. The simplest functional of this type on $D(R^n)$ is that defined by a unit mass at the origin. It is known as a Dirac δ-function and denoted by $\delta(x)$. Obviously $(\delta, \varphi) = \varphi(0)$ for all $\varphi \in D(R^n)$. Another notation for this equality is

$$\int_{R^n} \delta(x)\varphi(x)dx = \varphi(0).$$

It can be shown that this functional cannot be represented in the form (0.5.1) with a locally integrable function $f(x)$. There are also continuous linear functionals which are not representable in either of the forms (0.5.1) or (0.5.2).

The set $D^*(G)$ of all continuous linear functionals on the space $D(G)$ is known as the *space of distributions.* Distributions defined by (0.5.1) with a locally integrable function[9] $f(x)$ are said to be *regular.* We shall identify them with their representing functions $f(x)$. The notation $f(x)$ will often be used for an arbitrary nonregular distribution, though, of course, the phrase "the value of a function at a point" is usually meaningless for distributions. The value of an arbitrary functional $f \in D^*(G)$ on a test function $\varphi(x)$ will be denoted by (f, φ) or, in "integral" notation,

$$(f, \varphi) = \int_G f(x)\varphi(x)dx.$$

One also says that a distribution f *vanishes* in an open set $G_1 \subset G$ if $(f, \varphi) = 0$ for all $\varphi \in D(G_1)$. It can be shown that a distribution which vanishes in the neighborhood of each point of an open set will vanish on this set. The *support* of a distribution $f \in D^*(G_1)$ is the complement in G of the largest open set in which $f(x)$ vanishes. If the support of a distribution $f \in D^*(G)$ is relatively compact in G, then f may clearly be regarded as an element of any space $D^*(G')$, $G' \supset G$.[10] In other words, the functional f, originally defined on $D(G)$, may be extended to a continuous linear functional on any space $D(G')$ with $G' \supset G$.

Let $f_j \in D^*(G)$ be a sequence of distributions such that for some $f \in D^*(G)$ we have

$$\lim_{j\to\infty} (f_j, \varphi) = (f, \varphi) \qquad \forall \varphi \in D(G).$$

Then the sequence f_j is said to converge to the distribution f. Note that the space $D^*(G)$ is complete in the topology thus defined.

2. *Operations with distributions.* Let $f(x)$ be a differentiable function in a domain $G \subset R^n$, and let $\varphi(x) \in D(G)$. Since $\varphi(x)$ has compact support, it follows that for any $i = 1, \cdots, n$

[9] A necessary and sufficient condition for two locally integrable functions to define the same distribution by (0.5.1) is that they be equal almost everywhere.

[10] To define (f, φ) for $\varphi \in D(G')$, the function $\varphi(x)$ is expressed in some way as $\varphi(x) = \varphi_1(x) + \varphi_2(x)$, where $\varphi_1 \in D(G)$ and $\varphi_2(x) = 0$ in a neighborhood of supp f; one then sets $(f, \varphi) = (f, \varphi_1)$. A direct check shows that (f, φ) is well defined for $\varphi \in D(G')$.

$$\int_G \frac{\partial f}{\partial x_i}\,\varphi(x)dx = -\int_G \frac{\partial \varphi}{\partial x_i}\,f(x)dx. \tag{0.5.3}$$

This fact is used to define the derivative of a distribution. *The derivative with respect to x_i of a distribution $f \in D^*(G)$ is defined as the distribution $\partial f/\partial x_i$ whose value as a functional on every function $\varphi \in D(G)$ is defined by*

$$(\partial f/\partial x_i, \varphi) = -(f, \partial\varphi/\partial x_i).$$

By (0.5.3), if $f(x)$ is a continuously differentiable function, the distribution $\partial f/\partial x_i$ is the regular distribution defined by the ordinary function $\partial f/\partial x_i$.

Similarly, the distribution

$$\frac{\partial^{\|k\|} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \qquad (k = (k_1, \cdots, k_n))$$

is defined by the equality

$$\left(\frac{\partial^{\|k\|} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \varphi\right) = (-1)^{\|k\|}\left(f, \frac{\partial^{\|k\|}\varphi}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}\right) \qquad \forall \varphi \in D^*(G).$$

Thus every distribution has derivatives of arbitrary order. Note that the operator

$$D^k = \frac{\partial^{\|k\|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

is a continuous linear operator in $D^*(G)$. Indeed, if f_j is a sequence of distributions which converges in $D(G)$ to a distribution $f(x)$, then

$$\lim_{j\to\infty}(D^k f_j, \varphi) = (-1)^{\|k\|}\lim_{j\to\infty}(f_j, D^k\varphi)$$

$$= (-1)^{\|k\|}(f, D^k\varphi) = (D^k f, \varphi) \qquad \forall \varphi \in D(G)$$

and thus the operator D^k is continuous. Linearity is obvious. Note moreover that $\operatorname{supp} D^k f \subset \operatorname{supp} f$.

We now consider the *multiplication of a distribution and an infinitely differentiable function.* Let $f(x) \in D^*(G)$ and let $\psi(x)$ be an infinitely differentiable function in the domain G. We define the product ψf to be the distribution whose value as a functional on any test function is defined by

$$(\psi f, \varphi) = (f, \psi\varphi).$$

The right-hand side of this relation is meaningful, since, as is readily seen, $\psi\varphi$ is a test function. It is obvious that for regular distributions this definition is consistent with the usual sense of the product ψf. It is also obvious that the operator defined by multiplication by an infinitely differentiable function is continuous and linear in $D^*(G)$.

The last operation considered here for distributions is convolution.[11] Let $\psi(x)$ be a distribution in $D^*(R^n)$ whose support is concentrated in the ball $E_\varepsilon = \{x: |x| < \varepsilon\}$, and G_1 a domain whose ε-neighborhood G_1^ε is relatively compact in G. We define the convolution of $\psi(x)$ with a test function $\varphi(x) \in D(G_1) \subset D(R^n)$ to be the function

$$\psi * \varphi = (\psi(x), \varphi(x + y)) = \int_{R^n} \psi(x)\varphi(x + y)\, dx.$$

This function is clearly everywhere defined in $R^n_{(y)}$. It is also readily seen that $\psi * \varphi \in D(G)$. Indeed,

$$\frac{\partial(\psi * \varphi)}{\partial y_1} = \lim_{\Delta y_1 \to 0} \frac{\Delta_{y_1}(\psi * \varphi)}{\Delta y_1} = \lim_{\Delta y_1 \to 0} \left(\psi(x), \frac{\Delta_{y_1}\varphi(x + y)}{\Delta y_1}\right), \tag{0.5.4}$$

where

$$\Delta_{y_1}\varphi(x + y) = \varphi(x_1 + y_1 + \Delta y_1, x_2 + y_2, \cdots, x_n + y_n) - \varphi(x + y).$$

For any fixed $y \in G_1$, the functions $(1/\Delta y_1)\Delta_{y_1}\varphi(x + y)$ obviously converge as $\Delta y_1 \to 0$ in the space $D(R^n)$ to the function

$$\frac{\partial}{\partial y_1}\varphi(x + y) = \frac{\partial}{\partial x_1}\varphi(x + y).$$

Therefore

$$\lim_{\Delta y_1 \to 0} \left(\psi(x), \frac{1}{\Delta y_1}\Delta_{y_1}\varphi(x + y)\right) = \left(\psi(x), \frac{\partial}{\partial x_1}\varphi(x + y)\right)$$

$$= -\left(\frac{\partial \psi}{\partial x_1}, \varphi(x + y)\right) = -\frac{\partial \psi}{\partial x_1} * \varphi. \tag{0.5.5}$$

It follows from (0.5.4) and (0.5.5.) that the first derivatives of the function $\psi * \varphi$ exist and

$$\frac{\partial}{\partial y_i}(\psi * \varphi) = \psi * \frac{\partial \varphi}{\partial x_i} = -\frac{\partial \psi}{\partial x_i} * \varphi.$$

Since the first derivatives of $\psi * \varphi$ are again convolutions of a distribution and a test function, it follows that $\psi * \varphi$ has derivatives of arbitrary order. The fact that $\psi * \varphi$ has compact support in the domain G is obvious. Consequently $\psi * \varphi \in D(G)$.

We now define the convolution $f * \psi$ of the above distribution ψ with an arbitrary distribution $f(x) \in D^*(G)$ as the distribution on $D(G_1)$ whose value on each function $\varphi \in D(G_1)$ is defined by

$$(f * \psi, \varphi) = (f, \psi * \varphi).$$

We also set $\psi * f = f * \psi$; it is easily verified that this involves no contradictions.

[11] This is not the most general definition of convolution, but it is sufficient for our purposes.

Note, moreover, that the convolution operation is associative, provided, of course, that all distributions in the convolution, except one, have supports lying in suitable balls E_ε with sufficiently small ε.

We now show that if the distributions $\psi \in D^*(R^n)$ and $f \in D^*(R^n)$ are regular our definition of convolution coincides with the ordinary definition. Indeed, if $\varphi \in D(R^n)$, then

$$\begin{aligned}(f, \psi * \varphi) &= \int_{R^n} f(y)\left\{\int_{R^n} \varphi(x + y)\psi(x)\, dx\right\} dy \\ &= \int_{R^n} f(y)\left\{\int_{R^n} \varphi(\xi)\psi(\xi - y)\, d\xi\right\} dy \\ &= \int_{R^n} \varphi(\xi)\left\{\int_{R^n} f(y)\psi(\xi - y) dy\right\} d\xi = (f * \psi, \varphi).\end{aligned}$$

The convolution $f * \psi$ in the last term is the ordinary (classical) convolution of the functions $f(x)$ and $\psi(x)$.

The convolution $f * \psi$ is differentiated according to the rule

$$\frac{\partial}{\partial y_i}(f * \psi) = \frac{\partial f}{\partial x_i} * \psi.$$

Indeed,

$$\begin{aligned}\left(\frac{\partial}{\partial y_i}(f * \psi), \varphi\right) &= -\left(f * \psi, \frac{\partial \varphi}{\partial y_i}\right) = -\left(f, \psi * \frac{\partial \varphi}{\partial y_i}\right) \\ &= -\left(f, \frac{\partial}{\partial x_i}(\psi * \varphi)\right) = \left(\frac{\partial f}{\partial x_i}, \psi * \varphi\right) = \left(\frac{\partial f}{\partial x_i} * \psi, \varphi\right).\end{aligned}$$

If $\psi(x)$ is a regular infinitely differentiable distribution the convolution $f * \psi$ is also regular and infinitely differentiable (in the ordinary sense). Indeed, if $\varphi \in D(G_1)$, it follows from the linearity and continuity of the functional $f(x)$ that

$$\begin{aligned}(f * \psi, \varphi) = (f, \psi * \varphi) &= \left(f, \int_{|y| < \varepsilon} \varphi(x + y)\psi(y)\, dy\right) \\ &= \left(f(x), \int_{G_1} \varphi(y)\psi(y - x) dy\right) = \int_{G_1} \varphi(y)(f(x), \psi(y - x))\, dy.\end{aligned}$$

As was shown above, the function $(f(x), \psi(y - x))$ is infinitely differentiable in the domain G_1. This proves that in this case the functional $f * \psi$ is regular and is defined by the infinitely differentiable function $(f(x), \psi(y - x))$.

3. *Positive distributions and measures.* A distribution $f(x) \in D^*(G)$ is said to be *positive* if $(f, \varphi) \geqq 0$ for any nonnegative test function $\varphi(x) \in D(G)$.

It is readily seen that any continuous function $f(x)$ is positive in this sense (as a regular distribution) if and only if it is nonnegative (in the classical sense).

We noted above that every measure μ in the domain G defines a functional in $D^*(G)$ whose value (μ, φ) on functions $\varphi \in D(G)$ is

$$(\mu, \varphi) = \int_G \varphi(x) d\mu(x).$$

It is obvious that $(\mu, \varphi) \geqq 0$ for any nonnegative test function, and hence the functional defined by the measure μ is a positive distribution. It is proved in the theory of distributions[12] that the converse is also true:

With every positive distribution $f(x) \in D^*(G)$ one can associate a unique measure μ_f in the domain G such that

$$(f, \varphi) = \int_G \varphi(x) d\mu_f(x).$$

Thanks to this representation, the functional $f(x)$ can be extended from the space $D(G)$ to the space of all measurable functions $\varphi(x)$ with compact support in G which are continuous on the set $\{x: \varphi(x) \neq 0\}$. We define the integral of the product of a positive distribution and a function $\varphi(x)$ in this larger space:

$$\int_G f(x)\varphi(x)dx = \int_G \varphi(x) d\mu_f(x).$$

In particular, for a Borel set $E_1 \Subset G$ we set

$$\int_{E_1} f(x)dx = \int_{E_1} d\mu_f(x) = \mu_f(E_1).$$

Note that if μ_f is the measure associated with a positive distribution $f(x)$, the measure of an open set $G_1 \Subset G$ may also be defined by

$$\mu_f(G_1) = \lim_{j\to\infty}(f, \varphi_j), \tag{0.5.6}$$

where the test functions $\varphi_j(x)$ satisfy $\varphi_j(x) \leqq \varphi_{j+1}(x)$ $\forall x \in G$, $\operatorname{supp}\varphi_j \subset G_1$, $0 \leqq \varphi_j(x) \leqq 1$ $\forall x \in G_1$, and the sets $K_j = \{x: \varphi_j(x) = 1\}$ form a monotone increasing sequence of compact sets whose union is the domain G_1. Indeed,

$$(f, \varphi_j) = \int_{G_1} \varphi_j d\mu_f = \int_{K_j} d\mu_f + \int_{G_1\setminus K_j} \varphi_j d\mu_f = \mu_f(K_j) + \gamma_j, \tag{0.5.7}$$

where $0 \leqq \gamma_j \leqq \mu_f(G_1\setminus K_j)$. Since $\bigcup_1^\infty K_j = G_1$ and $K_j \subset K_{j+1}$ for all j, it follows that

$$\lim_{j\to\infty} \mu_f(K_j) = \mu_f(G_1) \qquad \text{and} \qquad \lim_{j\to\infty} \mu_f(G_1\setminus K_j) = 0.$$

Hence, via (0.5.7), we get (0.5.6).

Similarly, the measure $\mu_f(K)$ of a compact set $K \subset G$ may be found from

$$\mu_f(K) = \lim_{j\to\infty}(f, \psi_j), \tag{0.5.8}$$

where the test functions ψ_j satisfy the conditions

[12] See, for example, Schwartz [1], Gel'fand and Šilov [1], or Landkof [1].

$$\psi_j(x) \geqq \psi_{j+1}(x) \qquad \forall x \in G,$$
$$\psi_j(x) = 1 \qquad \forall x \in K,$$
$$0 \leqq \psi_j(x) \leqq 1 \qquad \forall x \in G$$

and

$$\bigcap_{j=1}^{\infty} \operatorname{supp} \psi_j = K.$$

We now consider a sequence of positive distributions $f_j(x)$ which converges in $D^*(G)$ to a distribution $f(x)$, which is also clearly positive. Then, for any open set $G_1 \Subset G$ and any compact set $K \subset G$,

$$\overline{\lim_{j\to\infty}} \int_K f_j(x)dx \leqq \int_K f(x)dx, \tag{0.5.9}$$

$$\underline{\lim_{j\to\infty}} \int_{G_1} f_j(x)dx \geqq \int_{G_1} f(x)dx.^{13)} \tag{0.5.10}$$

To prove these inequalities, let φ_j and ψ_j be the sequences of test functions figuring in (0.5.6) and (0.5.8) for the domain G_1 and the compact set K. It is clear that, for any i and j,

$$\int_K f_j(x)dx \leqq \int_G f_j\psi_i dx.$$

Consequently

$$\overline{\lim_{j\to\infty}} \int_K f_j(x)dx \leqq \int_G f(x)\psi_i(x)dx,$$

and hence by (0.5.8) it follows that

$$\overline{\lim_{j\to\infty}} \int_K f_j(x)dx \leqq \int_K f(x)dx.$$

Inequality (0.5.10) is proved in a similar manner.

If $\int_{\partial G_1} f(x)dx = 0$, and therefore $\int_{G_1} f(x)\,dx = \int_{\bar{G}_1} f(x)dx$, it follows immediately from (0.5.9) and (0.5.10) that

$$\lim_{j\to\infty} \int_{\bar{G}_1} f_j(x)dx = \lim_{j\to\infty} \int_{G_1} f_j(x)dx = \int_{G_1} f(x)dx.$$

§ 6. Multiple power series

Though the reader is assumed to be familiar with the elements of the theory of holomorphic functions of several variables, we nevertheless present here some information on multiple power series.

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13) An obvious alternative notation for these inequalities is

$$\overline{\lim_{j\to\infty}}\, \mu_{f_j}(K) \leqq \mu_f(K), \qquad \underline{\lim_{j\to\infty}}\, \mu_{f_j}(G_1) \geqq \mu_f(G_1).$$

We first recall that a function $f(z)$, $z \in C^n$, is said to be holomorphic or analytic at a point $z^0 \in C^n$ if it can be expanded in some neighborhood of z^0 as an absolutely convergent power series:

$$f(z) = \sum_{\|k\|=0}^{\infty} a_k (z - z^0)^k,$$

A function $f(z)$ is said to be holomorphic or analytic in a domain G (on a set M) if it is holomorphic at each point of the domain (set).

It is easy to see that a function $f(z)$ which is analytic in a circular polycylinder[14] $\mathscr{E}(r, z^0) = \{z: |z_i - z_i^0| < r_i, i = 1, \cdots, n\}$ can be expanded there as an absolutely convergent power series

$$f(z) = \sum_{\|k\|=0}^{\infty} a_k (z - z^0)^k;$$

this expansion is unique, and

$$a_k = \frac{1}{k!} \frac{\partial^{\|k\|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}\bigg|_{z=z_0}.$$

If, moreover, there exists $M < \infty$ such that $|f(z)| \leqq M$ for all $z \in \mathscr{E}(r, z^0)$, then, for any k,

$$|a_k| \leqq \frac{M}{r^k} = \frac{M}{r_1^{k_1} \cdots r_n^{k_n}}. \tag{0.6.1}$$

We now introduce a concept which arises naturally in the context of multiple power series: the concept of a multicircular domain.

DEFINITION. A domain $G \subset C^n$ is said to be *multicircular* with center at z^0 if, together with every point z', G contains any point z whose coordinates z_i satisfy the conditions

$$|z_i - z_i^0| = |z_i' - z_i^0| \qquad (i = 1, \cdots, n).$$

A multicircular domain $G \subset C^n$ with center at z^0 is said to be *complete* if, together with each point $z' \in G$, it contains the entire polydisk

$$\{z: |z_i - z_i^0| \leqq |z_i' - z_i^0|, i = 1, \cdots, n\}.$$

Let $G \subset C^n$ be a multicircular domain. We denote by $|G|$ the image in R_+^n of the domain G under the mapping $r_i = |z_i - z_i^0|$, $i = 1, \cdots, n$. We shall call this set

[14] Recall that a domain $D \subset C^n$ is said to be polycylindrical, or simply a polycylinder, if it has the form $D = \{z: z_j \in D_i, i = 1, \cdots, n\}$, where D_i is a domain in C^1. The *distinguished boundary* of the polycylinder is the set $\{z: z_i \in \partial D_i, = 1, \cdots, n\}$. A polycylindrical domain D is said to be a circular polycylinder, or simply a polydisk, if all the domains D_i are disks.

the image of the multicircular domain in the "absolute hyperoctant" (in the "absolute quadrant" if $n = 2$). A set $E \subset R_+^n$ is called a complete domain in R_+^n if, together with each point $r^0 \in E$, it contains all points r such that $0 \leqq r_i \leqq r_i^0$ and it contains no points of the closure of its complement in R_+^n.

It is obvious that the image of a complete multicircular domain in the absolute hyperoctant is a complete domain there.

Now consider the multiple power series

$$f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k.^{15)} \tag{0.6.2}$$

The *convergence domain* of this series is the interior D of the set of points z at which this series is absolutely convergent. It is clear that the convergence domain of the series (0.6.2) is a complete multicircular domain. We claim that it is also logarithmically convex, i.e. the image $\ln |D|$ of the set $|D|$ under the mapping $t_i = \ln r_i$, $i = 1, \cdots, n$, is a convex domain in $R_{(t)}^n$. Indeed, let $t' \in \ln |D|$ and $t'' \in \ln |D|$. Then

$$\sum_{\|k\|=0}^{\infty} |a_k| \exp\left\{\sum_{i=1}^{n} k_i t_i'\right\} < \infty, \qquad \sum_{\|k\|=0}^{\infty} |a_k| \exp\left\{\sum_{i=1}^{n} k_i t_i''\right\} < \infty.$$

Hence, using the obvious inequality $A + B > A^\alpha B^\beta$, where $A, B \geqq 0$, $\alpha, \beta \geqq 0$, and $\alpha + \beta = 1$, we conclude that

$$\sum_{\|k\|=0}^{\infty} |a_k| \exp\left\{\sum_{i=1}^{n} k_i(\alpha t_i' + \beta t_i'')\right\} < \infty.$$

Consequently any point t with coordinates $t_i = \alpha t_i' + \beta t_i''$, i.e. any point of the segment joining t' and t'', lies in the domain $\ln |D|$, which is therefore convex.

As we mentioned above, the set $|D|$ is a complete domain in R_+^n, and it is therefore uniquely determined by that part of its boundary lying within R_+^n. The closure of this part of the boundary of $|D|$ is known as the *hypersurface of associated radii of convergence* of the series (0.6.2). The numbers $r_1, \cdots, r_n$ (the coordinates of some point of this hypersurface) are known as associated radii of convergence. It is readily seen that positive numbers $r_1, \cdots, r_n$ are associated radii of convergence of the series (0.6.2) if and only if

$$\overline{\lim_{\|k\| \to \infty}} (|a_k| r^k)^{1/\|k\|} = 1. \tag{0.6.3}$$

Now assume that $|f(z)| \leqq M$ for all $z \in D$. Minimizing inequalities (0.6.1) over all polydisks contained in D, we get the following estimates (Cauchy's inequalities) for the coefficients of the power series (0.6.2):

$$|a_k| \leqq M \exp(-H_{\ln|D|}(k)) = M \exp(-|k| H_{\ln|D|}(k^0)), \tag{0.6.4}$$

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15) We have set $z^0 = 0$ only to simplify the notation.

where $k_0 = k/|k|$ and $H_{\ln|D|}(k)$ is the support function of the domain $\ln|D|$.

We also mention that every function holomorphic in a complete multicircular domain has a unique expansion there as an absolutely convergent multiple power series.

Later, together with the series (0.6.2), we shall consider series

$$\sum_{\|k\|=0}^{\infty} \frac{a_k}{z^k}. \tag{0.6.5}$$

The convergence domain of a series of this type is defined like that of the series (0.6.2). The boundary of the set $|D|$, where D is the convergence domain of the series (0.6.5), is called the hypersurface of associated radii of convergence of the series. Numbers $r_1, \cdots, r_n$ which are the coordinates of some point of this hypersurface are known as associated radii of convergence of the series (0.6.5). It follows from (0.6.3) that positive numbers $r_1, \cdots, r_n$ are associated radii of convergence of the series (0.6.5) if and only if

$$\lim_{\|k\|\to\infty} \left(\frac{|a_k|}{r^k}\right)^{1/\|k\|} = 1. \tag{0.6.6}$$

CHAPTER 1

SUBHARMONIC FUNCTIONS

§ 1. Elementary theory of subharmonic functions

1. *Definition of subharmonic functions. Examples.* Let $u(x)$, $x \in R^m$, be a function with values in the interval $[-\infty, \infty)$, measurable and bounded above on the sphere

$$S_r(x_0) = \{x: |x - x_0| = r\}.$$

Let $\mathfrak{M}_u(x_0, r)$ denote the average of the function $u(x)$ on the sphere $S_r(x_0)$:

$$\mathfrak{M}_u(x_0, r) = \frac{1}{\sigma_m r^{m-1}} \int_{S_r(x_0)} u(x) d\sigma, \tag{1.1.1}$$

where

$$\sigma_m = \frac{m\pi^{m/2}}{\Gamma(m/2 + 1)}$$

is the area of the unit sphere in R^m, and $d\sigma$ is the area element on the sphere $S_r(x_0)$. Similarly, if the function $u(x)$ is measurable and bounded above in the ball

$$E_r(x_0) = \{x: |x - x_0| < r\},$$

we let $\mathfrak{N}_u(x_0, r)$ denote the average of $u(x)$ in the ball $E_r(x_0)$:

$$\mathfrak{N}_u(x_0, r) = \frac{1}{\omega_m r^m} \int_{E_r(x_0)} u(x) d\omega, \tag{1.1.2}$$

where

$$\omega_m = \frac{\pi^{m/2}}{\Gamma(m/2 + 1)}$$

is the volume of the ball $E_1(0)$, and $d\omega$ is the volume element in the space R^m.

In view of our assumptions on the function $u(x)$, the quantities $\mathfrak{M}$ and $\mathfrak{N}$ assume values in the interval $[-\infty, \infty)$.[1] Note also that they are related as follows:

[1] When the integrals figuring in (1.1.1) and (1.1.2) do not exist in the ordinary sense, it is natural to assign them the value $-\infty$ (since the function $u(x)$ is bounded above).

$$\mathfrak{N}_u(x_0, r) = \frac{m}{r^m} \int_0^r \mathfrak{M}_u(x_0, t) t^{m-1}\, dt. \tag{1.1.3}$$

It is well known that if the function $u(x)$ is a harmonic function, i.e. a solution of the Laplace equation $\Delta u = 0$, then

$$\mathfrak{M}_u(x, r) = u(x). \tag{1.1.4}$$

Moreover, it is known that this equality is a necessary and sufficient condition for a function to be harmonic.

A class of functions closely related to harmonic functions is obtained by replacing the equality sign in (1.1.4) by an inequality sign.

DEFINITION. A function $u(x)$ defined in a domain $G \subset R^m$ is said to be *subharmonic* in G if it is upper semicontinuous in G and, for any point $x \in G$ and all sufficiently small positive r,

$$u(x) \leqq \mathfrak{M}_u(x, r).^{2)} \tag{1.1.5}$$

It follows from the definition that any harmonic function is also subharmonic. For $m > 2$ the function $u(x) = -|x|^{2-m}$ is also subharmonic. Indeed, it is easily verified that this function is harmonic everywhere except at the origin. Thus, at every point $x \neq 0$ inequality (1.1.5) holds for all $r \in (0, |x|)$. It is obvious that inequality (1.1.5) holds at $x = 0$. It is also obvious that this function is upper semicontinuous.

Other examples of subharmonic functions are provided by the theory of analytic functions. Thus, if the function $g(z)$, $z \in C^1$, is analytic in a domain $G \subset C^1$, then the functions $\ln|g(z)|$ and $|g(z)|^\alpha$, $\alpha > 0$, are subharmonic in G. The proof that $\ln|g(z)|$ is subharmonic is exactly the same as for the function $-|x|^{2-m}$. To prove that $|g(z)|^\alpha$ is subharmonic, it is obviously sufficient to show that condition (1.1.5) holds. This is clear for all z such that $g(z) = 0$. But if $g(z) \neq 0$, then for a sufficiently small neighborhood of the point z we can take a univalent analytic branch of the function $g^\alpha(z)$, and then

$$|g(z)|^\alpha = |g^\alpha(z)| = \frac{1}{2\pi}\left|\int_0^{2\pi} g^\alpha(z + re^{i\varphi})\, d\varphi\right| \leqq \frac{1}{2\pi}\int_0^{2\pi} |g(z + re^{i\varphi})|^\alpha d\varphi = \mathfrak{M}_{|g|^\alpha}(z, r).$$

Later we shall prove that the functions $|g(z)|^\alpha$ and $\ln|g(z)|$ are also subharmonic when $z \in C^n$, i.e. when $g(z)$ is an analytic function of m complex variables $z_1, \cdots, z_m$.

2. *Simple properties of subharmonic functions.* The following assertions are obvious.

(a) *The product of a subharmonic function and a positive constant is again a subharmonic function.*

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2) It seems to us convenient to include $u(x) \equiv -\infty$ in the class of subharmonic functions; this convention is not universally adopted.

(b) *The sum of finitely many subharmonic functions is again a subharmonic function.*

(c) *If the functions* $u_1(x), \cdots, u_N(x)$ *are subharmonic in a domain* $G \subset R^m$, *then the function* $u(x) = \max_{1 \leqq j \leqq N} u_j(x)$ *is also subharmonic in* G.

(d) *The limit of a uniformly convergent sequence of subharmonic functions is a subharmonic function.*

(e) *The limit of a monotone decreasing sequence of subharmonic functions is a subharmonic function.*

REMARK. Properties (d) and (e) are also valid when instead of a sequence of subharmonic functions one considers a family of subharmonic functions $u_t(x)$ depending upon a continuous parameter t.

3. *The maximum principle. Harmonic majorant.* The following property of subharmonic functions is known as the "maximum principle."

If the function $u(x)$ *is subharmonic in a domain* $G \Subset R^m$ *and* $u(x) \not\equiv \text{const}$, *then*

$$u(x) < \sup_{x' \in \partial G} \overline{\lim_{\substack{x'' \to x' \\ x'' \in G}}} u(x'') \qquad \forall x \in G.$$

Indeed, let $M = \sup_G u(x)$. If $M = +\infty$, there exists a sequence of points $x_k \in G$, $k = 1, 2, \cdots$, such that $\lim_{k \to \infty} u(x_k) = +\infty$. Since $\bar{G}$ is compact, we may assume that this sequence converges to a point $x_0 \in \bar{G}$. Since the function $u(x)$ is upper semicontinuous, it is bounded above on any compact subset $K \subset G$. Therefore $x_0 \notin G$, and hence $x_0 \in \partial G$. Consequently, when $M = +\infty$,

$$\sup_{x' \in \partial G} \overline{\lim_{\substack{x'' \to x' \\ x'' \in G}}} u(x'') = +\infty,$$

while $u(x) < +\infty$ $\forall x \in G$. This proves the maximum principle for the case $M = +\infty$.

Now let $M < +\infty$. Consider the set E of all points $x \in G$ at which $u(x) = M$. Since the function $u(x)$ is semicontinuous and not identically constant, E is a relatively closed proper subset of the domain G. Thus the assumption $E \neq \emptyset$ implies that there exists a point $x' \in E$ with arbitrarily small neighborhoods containing points not in E. But then there must be a sequence of numbers r_j such that $r_j \to \infty$ as $j \to \infty$ and $\mathfrak{M}_u(x', r_j) < M$. Since $u(x') = M$, this contradicts the subharmonicity of $u(x)$. Consequently $E = \emptyset$, and so $u(x) < M$ for all $x \in G$. To complete the proof it now suffices to repeat the argument used in the case $M = +\infty$.

The maximum principle implies the following property of subharmonic functions:

Let the function $u(x)$ *be subharmonic in the domain* $G \Subset R^m$, *and let the function* $v(x)$ *be harmonic in* G *and continuous in* $\bar{G}$. *Then, if the inequality*

$$\overline{\lim_{\substack{x' \to x \\ x' \in G}}} u(x) \leqq v(x)$$

holds at all boundary points of G, *it must hold at all interior points too. In this case, if*

there is at least one point $x_0 \in G$ such that $u(x_0) = v(x_0)$, then $u(x) \equiv v(x)$ in G.

Indeed, consider the function $\Phi(x) = u(x) - v(x)$. Since the function $-v(x)$, being harmonic, is therefore also subharmonic, it follows from property (b) of § 1.2 that $\Phi(x)$ is also subharmonic. For $x \in \partial G$ we have

$$\overline{\lim_{x' \to x, x' \in G}} \Phi(x') \leqq 0.$$

Hence it follows by the maximum principle that $\Phi(x) \leqq 0$ throughout G, and equality can hold only if $\Phi(x) \equiv 0$. Consequently $u(x) \leqq v(x)$ for all $x \in G$, and equality can hold only if $u(x) \equiv v(x)$.

This property, sometimes known as the *principle of the harmonic majorant*, is characteristic for subharmonic functions. To be precise, the following assertion is valid:

Let the function $u(x)$ be upper semicontinuous in a domain G. Assume further that for any domain $D \Subset G$ and any function $v(x)$ harmonic in D and continuous in $\bar{D}$ the inequality

$$\overline{\lim_{x' \to x, x' \in D}} u(x') \leqq v(x) \qquad \forall x \in \partial D$$

implies the inequality $u(x) \leqq v(x)$ for all $x \in D$. The $u(x)$ is a subharmonic function.

For the proof it is sufficient to verify inequality (1.1.5). Let $x_0 \in G$ and take $r > 0$ so small that the ball $E_r(x_0)$ is relatively compact in G. Since $u(x)$ is upper semicontinuous, it may be expressed as $u(x) = \lim_{j \to \infty} u_j(x)$, where $\{u_j(x)\}$ is a monotone decreasing sequence of continuous functions. Construct a sequence of functions $v_j(x)$, each harmonic in $E_r(x_0)$ and continuous in $\bar{E}_r(x_0)$, such that $v_j(x) = u_j(x)$ for all $x \in S_r(x_0)$. The existence of the functions $v_j(x)$ follows from the solvability of the Dirichlet problem for a ball. Since $u(x) \leqq u_j(x)$, it follows that

$$\overline{\lim_{\substack{x' \to x \\ x' \in E_r(x_0)}}} u(x') \leqq v_j(x)$$

for $x \in S_r(x_0)$. Consequently $u(x) \leqq v_j(x)$ for all $x \in E_r(x_0)$, and in particular

$$u(x_0) \leqq v_j(x_0) \qquad (j = 1, 2, \cdots). \tag{1.1.6}$$

We now observe that, since $v_j(x)$ is harmonic,

$$v_j(x_0) = \mathfrak{M}_{v_j}(x_0, r).$$

Hence by (1.1.6) it follows that

$$u(x_0) \leqq \mathfrak{M}_{u_j}(x_0, r) \qquad (j = 1,2, \cdots). \tag{1.1.7}$$

Since the functions $u_j(x)$ form a monotone decreasing sequence, it follows that in the formula for the function $\mathfrak{M}_{u_j}(x_0, r)$ one can pass to the limit under the integral sign. Therefore, letting $j \to \infty$ in (1.1.7) we see that $u(x_0) \leqq \mathfrak{M}_u(x_0, r)$, whence it follows that $u(x)$ is subharmonic.

Subharmonic functions are often defined in terms of the principle of the harmonic majorant, rather than inequality (1.1.5).

4. *Properties of the functions* $\mathfrak{M}_u(x,r)$ *and* $\mathfrak{N}_u(x, r)$. The following assertions hold:

(a) *If the function* $u(x)$ *is subharmonic in a domain* $G \subset R^m$, *then the functions* $\mathfrak{M}_u(x, r)$ *and* $\mathfrak{N}_u(x, r)$ *are monotone nondecreasing functions of* r *in every interval* $(0,R)$, *where* $R \in \{R\colon E_R(x) \subset G\}$.

To prove this, we proceed as in § 1.3 and consider a monotone decreasing sequence of continuous functions $u_j(x)$ converging to $u(x)$, and a sequence of functions $v_j(x)$, harmonic in a ball $E_r(x_0) \Subset G$ and continuous in $\bar{E}_r(x_0)$, such that $v_j(x) = u_j(x)$ for all $x \in S_r(x_0)$. By the maximum principle, the inequality $u(x) \leqq v_j(x)$ holds throughout $E_r(x_0)$. Consequently $\mathfrak{M}_u(x_0, r') \leqq \mathfrak{M}_{v_j}(x_0, r')$ for $r' < r$. Since $v_j(x)$ is a harmonic function, $\mathfrak{M}_{v_j}(x_0, r') = \mathfrak{M}_{v_j}(x_0, r)$. Now it follows from the definition of $v_j(x)$ that $\mathfrak{M}_{v_j}(x_0, r) = \mathfrak{M}_{u_j}(x_0, r)$. Combining these relations, we get

$$\mathfrak{M}_u(x_0, r') \leqq \mathfrak{M}_{u_j}(x_0, r).$$

Letting $j \to \infty$ in this inequality, we now see that, for any $r' \in (0, r)$,

$$\mathfrak{M}_u(x_0, r') \leqq \mathfrak{M}_u(x_0, r).$$

This shows that the function $\mathfrak{M}_u(x_0, r)$ is monotone. The analogous assertion for the function $\mathfrak{N}_u(x_0, r)$ follows in an obvious manner from (1.1.3), rewritten as follows:

$$\mathfrak{N}_u(x_0, r) = m \int_0^1 s^{m-1} \mathfrak{M}_u(x_0, rs)\, ds.$$

(b) *For any admissible value of* r,

$$u(x) \leqq \mathfrak{N}_u(x, r) \leqq \mathfrak{M}_u(x, r). \tag{1.1.8}$$

To prove these inequalities we need only refer to (1.1.5) and (1.1.3) and use the monotonicity of $\mathfrak{M}_u(x_0, r)$.

(c) *For each point* $x \in G$,

$$\lim_{r \to 0} \mathfrak{M}_u(x, r) = \lim_{r \to 0} \mathfrak{N}_u(x, r) = u(x).$$

Indeed, it follows from (1.1.8) that

$$u(x) \leqq \lim_{r \to 0} \mathfrak{N}_u(x, r) \leqq \lim_{r \to 0} \mathfrak{M}_u(x, r). \tag{1.1.9}$$

On the other hand, the function $u(x)$ is upper semicontinuous, and therefore for any number $A > u(x)$ and all sufficiently small positive r, we have $\mathfrak{M}_u(x, r) < A$. Consequently

$$\lim_{r \to 0} \mathfrak{M}_u(x, r) \leqq u(x). \tag{1.1.10}$$

Combining (1.1.9) and (1.1.10), we see that

$$u(x) = \lim_{r \to 0} \mathfrak{N}_u(x, r) = \lim_{r \to 0} \mathfrak{M}_u(x, r).$$

(d) *Two subharmonic functions which are equal almost everywhere are identically equal.*

This is an obvious corollary of (c).

(e) *If $u(x) \not\equiv -\infty$ is subharmonic in a domain G, then $\mathfrak{N}_u(x, r) > -\infty$ for any r and x such that $E_{r_0}(x) \Subset G$.*

To prove this, suppose the assertion false, i.e. there are r_0 and x_0 such that $E_{r_0}(x_0) \Subset G$ but $\mathfrak{N}_u(x_0, r_0) = -\infty$. Then for any r and x such that $E_{r_0}(x_0) \subset E_r(x) \Subset G$ we have $\mathfrak{N}_u(x, r) = -\infty$. Thus, in view of (1.1.8), it follows that for some $\delta > 0$ the equality $u(x) \equiv -\infty$ holds everywhere in the ball $E_\delta(x_0)$. Thus, under our assumptions on the function $u(x)$, the set D of all points x such that $u(x) = -\infty$ has interior points. Denote the interior of D by D°. We shall show that the assumption $D^\circ \neq \emptyset$ implies $D^\circ = G$, contradicting the assumption $u(x) \not\equiv -\infty$. Indeed, suppose that $D^\circ \neq \emptyset$ and $D^\circ \neq G$. Then there exists a point $x' \in G$ which is a boundary point of D°. It is clear that for some $\eta > 0$ and all $x \in E_\eta(x')$ the ball $E_{2\eta}(x)$ is relatively compact in G and its intersection with the open set D° is not empty. Consequently $\mathfrak{N}_u(x, 2\eta) = -\infty$ for $x \in E_\eta(x')$, and in view of (1.1.8) it also follows that $u(x) = -\infty$ for $x \in E_\eta(x')$. But this is impossible, for x' is a boundary point of D. Thus $D^\circ = G$; and this, as already stated, contradicts our assumption concerning the function $u(x)$. This contradiction shows that $\mathfrak{N}_u(x, r) > -\infty$ whenever $E_r(x) \Subset G$.

REMARK 1. We have proved in passing that *the set on which a subharmonic function $u(x) \not\equiv -\infty$ is finite is dense in the domain.* This is often incorporated as a condition in the definition of subharmonic functions.[3)]

REMARK 2. Another, equivalent formulation of the assertion just proved is: *A subharmonic function is either locally integrable or identically equal to minus infinity.*

(f) *If $u(x) \not\equiv -\infty$ is subharmonic in a domain G, then $\mathfrak{M}_u(x, r) > -\infty$ for any r and x such that $E_r(x) \Subset G$.*

This assertion follows immediately from (e) and (1.1.8).

5. *Integration with respect to a parameter.* Consider domains $G \subset R^m_{(x)}$, $D \subset R^n_{(y)}$ and $D_0 \Subset D$, and let μ be a measure on D. Let $u(x, y)$ be a function defined and upper semicontinuous in the domain $G \times D$, subharmonic in x in the domain G for any fixed y in D. Then *the function*

$$v(x) = \int_{D_0} u(x, y)\, d\mu_y \tag{1.1.11}$$

is subharmonic in the domain G.

[3)] In this connection, see footnote 2 on the second page of this chapter.

Indeed, since the function $u(x, y)$ is upper semicontinuous, it may be expressed in any domain $G_0 \times D_0$, $G_0 \Subset G$, as a limit

$$u(x, y) = \lim_{j\to\infty} u_j(x, y),$$

where the functions $u_j(x, y)$ are continuous in $\bar{G}_0 \times \bar{D}_0$ and form a monotone decreasing sequence. From the functions $u_j(x, y)$ we derive functions

$$v_j(x) = \int_{D_0} u_j(x, y)\, d\mu_y.$$

The functions $v_j(x)$are clearly continuous in G_0, form a monotone decreasing sequence, and

$$\lim_{j\to\infty} v_j(x) = \lim_{j\to\infty} \int_{D_0} u_j(x, y)\, d\mu_y = \int_{D_0} u(x, y) d\mu_y = v(x).$$

Thus the function $v(x)$ is upper semicontinuous in G_0, and hence also in G. Note moreover that if $E_r(x_0) \Subset G_0$, then

$$\mathfrak{M}_{v_j}(x_0, r) = \frac{1}{r^{m-1}\sigma_m} \int_{S_r(x_0)} d\sigma_x \int_{D_0} u_j(x, y) d\mu_y$$

$$= \frac{1}{r^{m-1}\sigma_m} \int_{D_0} d\mu_y \int_{S_r(x_0)} u_j(x, y) d\sigma_x = \int_{D_0} \mathfrak{M}_{u_j}(x_0, r) d\mu_y.$$

Letting $j \to \infty$ in this equality and using the monotonicity of the sequences $\{u_j\}$ and $\{v_j\}$, we get

$$\mathfrak{M}_v(x_0, r) = \int_{D_0} \mathfrak{M}_u(x_0, r) d\mu_y.$$

Since the function $u(x, y)$ is subharmonic in x, it follows that

$$\int_{D_0} \mathfrak{M}_u(x_0, r) d\mu_y \geqq \int_{D_0} u(x_0, y) d\mu_y = v(x_0).$$

Consequently $v(x_0) \leqq \mathfrak{M}_v(x_0, r)$, and thus $v(x)$ is subharmonic in G_0; since G_0 is arbitrary, this conclusion is valid for the entire domain G.

REMARK. If the total measure μ of the domain D is finite, this assertion clearly remains valid if the domain D_0 is replaced in the integral (1.1.11) by D.

COROLLARY. *If the function $\Phi(x)$ is subharmonic in the domain G, then the functions $\mathfrak{M}_\Phi(x, r)$ and $\mathfrak{N}_\Phi(x, r)$ are subharmonic in the open set G^r consisting of those points $x \in G$ whose distance to the nearest point of the boundary of G exceeds r.*

To prove this, we need only apply the preceding result to the function $u(x,y) = \Phi(x + y)$ and the domain $D_0 = E_r(x_0)$ or $D_0 = S_r(x_0)$.

6. *Approximation of subharmonic functions. Let $u(x)$ be a function subharmonic in*

some domain $G \subset R^m$. Let $\{G_\varepsilon\}$ be a family of domains such that[4] $E_{2\varepsilon}(x) \subset G\ \forall x \in G_\varepsilon$, $G_{\varepsilon_1} \subset G_{\varepsilon_2}$ *for* $\varepsilon_2 < \varepsilon_1$ *and* $\bigcup_{0<\varepsilon<\delta} G_\varepsilon = G$. *Then there exists a family of functions* $\{u_\varepsilon(x)\}_{0<\varepsilon<\delta}$ *satisfying the following conditions*:

(a) *For every* ε, $u_\varepsilon(x)$ *is an infinitely differentiable subharmonic function in* G_ε.

(b) $u_{\varepsilon_1}(x) \geqq u_{\varepsilon_2}(x)$ *for* $\varepsilon_1 > \varepsilon_2$ *and* $x \in G_{\varepsilon_1}$.

(c) $\lim_{\varepsilon\to 0} u_\varepsilon(x) = u(x)$ *for all* $x \in G$.

To prove this assertion, we consider some function $\alpha(t)$ which is infinitely differentiable on the real axis and satisfies the conditions $\alpha(t) = 0$ for $t \notin (0,1)$, $\alpha(t) \geqq 0$ for $t \in (0,1)$, and

$$\int_0^1 t^{m-1}\alpha(t)dt = \frac{1}{\sigma_m}.$$

Then the functions $\alpha_\varepsilon(x) = \varepsilon^{-m}\alpha(|x|/\varepsilon)$ are infinitely differentiable in R^m and vanish everywhere outside the ball $E_\varepsilon(0)$. Set

$$u_\varepsilon(x) = \int_{E_\varepsilon(0)} u(x + x')\alpha_\varepsilon(x')d\omega_{x'}. \tag{1.1.12}$$

Since the function $u(x + x')$ is integrable in G_ε and the function $\alpha_\varepsilon(x)$ is infinitely differentiable in R^m, it follows that $u_\varepsilon(x)$ is infinitely differentiable in G_ε. By § 1.5 it is also subharmonic in G_ε. Thus the functions $u_\varepsilon(x)$ satisfy condition (a) of our assertion. To show that conditions (b) and (c) also hold, we slightly modify (1.1.12), as follows:

$$\begin{aligned} u_\varepsilon(x) &= \int_{E_\varepsilon(0)} u(x + x')\alpha_\varepsilon(x')d\omega_{x'} = \int_0^\varepsilon dt \int_{S_t(0)} \alpha_\varepsilon(x')u(x + x')d\sigma_{x'} \\ &= \varepsilon^{-m}\int_0^\varepsilon \left\{\alpha\left(\frac{t}{\varepsilon}\right)\int_{S_t(0)} u(x + x')d\sigma_{x'}\right\}dt \\ &= \frac{\sigma_m}{\varepsilon^m}\int_0^\varepsilon \alpha\left(\frac{t}{\varepsilon}\right)t^{m-1}\mathfrak{M}_u(x, t)dt = \sigma_m\int_0^1 \alpha(s)s^{m-1}\mathfrak{M}_u(x, \varepsilon s)ds. \end{aligned}$$

Hence, in view of the properties of the functions $\mathfrak{M}_u(x, t)$ and $\alpha(s)$, it follows that $u_{\varepsilon_1}(x) \leqq u_{\varepsilon_2}(x)$ for $\varepsilon_1 \leqq \varepsilon_2$, while

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = \sigma_m\int_0^1 s^{m-1}\alpha(s)\left\{\lim_{\varepsilon\to 0}\mathfrak{M}_u(x, \varepsilon s)\right\}ds = \sigma_m\left(\int_0^1 s^{m-1}\alpha(s)ds\right)u(x) = u(x).$$

This completes the proof.

7. *Hartogs' Lemma.* The following proposition, first proved for a special case[5] by Hartogs, is usually known as Hartogs' Lemma.

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[4] That such a family indeed exists for any domain G is obvious.

[5] Hartogs considered subharmonic functions in the space C^1 which can be expressed as $u(z) = \ln|g(z)|$ with an analytic function $g(z)$. The earliest appearance known to us of Hartogs' Lemma in the version presented here is in Lelong [1].

Let $\{u_t(x)\}_{0<t<1}$ be a family of functions, subharmonic and uniformly bounded above in a domain $G \subset R^m$, and assume that for every point $x \in G$

$$\overline{\lim_{t\to 1}}\, u_t(x) \leqq A.$$

Then for any number $\varepsilon > 0$ and any compact set $K \subset G$ there exists a positive number $\delta = \delta(K, \varepsilon)$ such that $u_t(x) \leqq A + \varepsilon$ for $t \in (\delta, 1)$ and $x \in K$.

PROOF. Suppose the assertion false. Then there exist a number $\varepsilon > 0$, points $x^{(j)} \in K$, $j = 1, 2, \cdots$, and numbers t_j such that $t_j \to 1$ as $j \to \infty$ and $u_{t_j}(x^{(j)}) \geqq A + \varepsilon$. Without loss of generality we may assume that $A \geqq 0$, so that the subharmonic functions $u_t^+(x) = \max\{0, u_t(x)\}$ satisfy the following conditions:

$$\overline{\lim_{t\to 1}}\, u_t^+(x) \leqq A, \tag{1.1.13}$$

$$u_{t_j}^+(x^{(j)}) \geqq A + \varepsilon, \tag{1.1.14}$$

$$\sup_{t\in(0,1)} \sup_{x\in K} u_t^+(x) = M < \infty. \tag{1.1.15}$$

By Fatou's Lemma[6] it follows from (1.1.13) and (1.1.15) that, if $E_r(x) \Subset G$,

$$\overline{\lim_{t\to 1}}\, \mathfrak{N}_{u_t^+}(x, r) \leqq \mathfrak{N}_{u^+}(x, r) \leqq A, \tag{1.1.16}$$

where

$$u^+(x) = \overline{\lim_{t\to 1}}\, u_t^+(x).$$

It is now quite easy to derive a contradiction. Indeed, without loss of generality we may assume that the sequence $x^{(j)}$ converges to some point $x' \in K$. The point x' is an interior point of G, and therefore $E_{2r_0} \Subset G$ for some $r_0 > 0$. Since the functions $u_{t_j}^+(x)$ are subharmonic, it follows from (1.1.14) that for any $j = 1,2, \cdots$,

$$\mathfrak{N}_{u_{t_j}^+}(x^{(j)}, r_0) \geqq A + \varepsilon.$$

Further, in view of the fact that the functions $u_{t_j}^+$ are nonnegative and the function $\mathfrak{N}_u(x, r)$ monotone, for $|x' - x^{(j)}| < r$ we see that

$$\begin{aligned} A + \varepsilon \leqq \mathfrak{N}_{u_{t_j}^+}(x^{(j)}, r_0) &\leqq \left(\frac{r_0 + |x' - x^{(j)}|}{r_0}\right)^m \mathfrak{N}_{u_{t_j}^+}(x', r_0 + |x' - x^{(j)}|) \\ &\leqq \left(\frac{r_0 + |x' - x^{(j)}|}{r_0}\right)^m \mathfrak{N}_{u_{t_j}^+}(x', 2r_0). \end{aligned}$$

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[6] Fatou's Lemma may be stated as follows:

If $\{f_t(x)\}_{0<t<1}$ is a family of functions measurable and uniformly bounded above in the domain G, then

$$\overline{\lim_{t\to 1}} \int_G f_t(x)\, d\omega_x \leqq \int_G \left\{\overline{\lim_{t\to 1}} f_t(x)\right\} d\omega_x.$$

See, for example, Privalov [1], p. 15.

Letting $j \to \infty$ in this inequality, we get

$$A + \varepsilon \leqq \overline{\lim_{j\to\infty}} \mathfrak{N}_{u_j}^{+}(x', 2r_0).$$

This contradicts (1.1.16), proving the lemma.

§ 2. Integral representation of a subharmonic function

1. *Associated measure.* Let $u(x)$ be a function, subharmonic in a domain $G \subset R^m$, such that $u(x) \not\equiv -\infty$. It was mentioned in § 1 that a function satisfying these conditions is locally integrable and may therefore be regarded as a regular distribution. As such, the function $u(x)$ has derivatives of arbitrary order. In particular the Laplacian Δu exists. The following property of Δu plays an essential role in the theory of subharmonic functions:

For any subharmonic function $u(x)$ such that $u(x) \not\equiv -\infty$ we have $\Delta u \geqq 0$, i.e. Δu is a positive distribution.

We first prove the assertion for twice continuously differentiable functions. In this case the function $u(x)$ may be expanded in a neighborhood of a point $x_0 = (x_1^0, \cdots, x_m^0) \in G$ as a series

$$u(x) = u(x_0) + \sum_{i=1}^{m} a_i(x_i - x_i^0) + \sum_{i,j=1}^{m} a_{i,j}(x_i - x_i^0)(x_j - x_j^0) + o(|x - x_0|^2).$$

It is readily seen that this implies

$$\mathfrak{M}_u(x_0, r) = u(x_0) + \frac{t^2}{m} \sum_{i=1}^{m} a_{i,i} + o(r^2).$$

Since $u(x)$ is subharmonic, we now see that for $r \to 0$

$$0 \leqq \mathfrak{M}_u(x_0, r) - u(x_0) = \frac{r^2}{m}\left\{\sum_{i=1}^{m} a_{i,i} + o(1)\right\}.$$

Consequently $\sum_1^m a_{i,i} \geqq 0$. But

$$\sum_{i=1}^{m} a_{i,i} = \frac{1}{2} \Delta u\Big|_{x=x_0}$$

and so $\Delta u \geqq 0$ for all $x \in G$.

Now let $u(x) \not\equiv -\infty$ be an arbitrary, i.e. not necessarily differentiable, subharmonic function. By § 1.6 this function may be expressed as

$$u(x) = \lim_{j\to\infty} u_j(x),$$

where $\{u_j(x)\}$ is a monotone decreasing sequence of infinitely differentiable subharmonic functions. Now the functions $u_j(x)$ satisfy the conditions of the case just treated, and therefore $\Delta u_j \geqq 0$, $j = 1,2, \cdots$. To prove the inequality $\Delta u \geqq 0$, it

will suffice to show that $(\Delta u, \varphi) \geqq 0$ for all $\varphi \in D(G)$, $\varphi \geqq 0$. Indeed,

$$(\Delta u, \varphi) = (u, \Delta\varphi) = \int_G \left\{\lim_{j\to\infty} u_j\right\} \Delta\varphi d\omega = \lim_{j\to\infty} \int_G u_j \Delta\varphi d\omega$$

$$= \lim_{j\to\infty} (u_j, \Delta\varphi) = \lim_{j\to\infty} (\Delta u_j, \varphi) \geqq 0.$$

This completes the proof.

REMARK. In the same way as we removed the original restriction that the subharmonic function in question be sufficiently smooth, we can remove the analogous restriction (existence of continuous second derivatives) in property (b) of convex functions (Introduction, § 3). When this is done, the approximation of an arbitrary convex function by infinitely differentiable convex functions follows in the same way as the corresponding property of subharmonic functions.

As we stated in the Introduction, every positive distribution is a measure. More precisely, with any positive distribution $f(x) \in D^*(G)$ we can associate a unique measure $\mu = f(x)$ in the domain D:

$$(f, \varphi) = \int_G \varphi d\mu \qquad \forall \varphi \in D(G).$$

Hence with every subharmonic function $u(x) \not\equiv -\infty$ we can associate a unique measure (mass distribution) $\mu = \mu_u$ in the domain, such that $\mu = \Delta u/\theta_m$, where

$$\theta_m = \begin{cases} (m-2)\sigma_m & \text{for } m > 2, \\ 2\pi & \text{for } m = 2. \end{cases}$$

We shall call this measure the *associated measure* (or *Riesz measure*) *of the subharmonic function* $u(x)$.

2. *Integral representation.* We first show that every measure μ on a domain $G \subset R^m$ such that $\mu(G) < \infty$ is associated with at least one subharmonic function $u(x)$ such that $\Delta u/\theta_m = \mu$. Set

$$h_*(r) = \begin{cases} \ln r & \text{for } m = 2, \\ -r^{2-m} & \text{for } m > 2, \end{cases}$$

and $h(x) = h_*(|x|)$. The function $h(x)$ is subharmonic (see § 1.1). Note also that $\Delta h = \theta_m \delta(x)$, i.e. $(\Delta h, \varphi) = \theta_m \varphi(0)$ for all $\varphi \in D(R^m)$. Indeed,

$$(\Delta h, \varphi) = (h, \Delta\varphi) = \int_{R^m} h\Delta\varphi d\omega = \int_{R^m \setminus E_r(0)} h\Delta\varphi d\omega + \int_{E_r(0)} h\Delta\varphi d\omega.$$

Now by Green's identity we get

$$(\Delta h, \varphi) = \int_{R^m \setminus E_r(0)} \varphi \Delta h d\omega + \int_{S_r(0)} h \frac{\partial\varphi}{\partial n} d\sigma - \int_{S_r(0)} \varphi \frac{\partial h}{\partial n} d\sigma + o(1)$$

$$= -\int_{S_r(0)} \varphi \frac{\partial h}{\partial n} d\sigma + o(1),$$

and since

$$\frac{\partial h}{\partial n} = -\frac{\theta_m}{\sigma_m}\frac{1}{|x|^{m-1}},$$

it follows that

$$\int_{S_r(0)} \varphi \frac{\partial h}{\partial n} d\sigma = \theta_m \varphi(0) + o(1). \tag{1.2.1}$$

From (1.2.1) we conclude that $(\Delta h, \varphi) = \theta_m\varphi(0) + o(1)$; and since $(\Delta h, \varphi)$ is independent of r, we have $(\Delta h, \varphi) = \theta_m\varphi(0)$. Consequently

$$\Delta h = \theta_m \delta(x). \tag{1.2.2}$$

Using this fact, one readily sees that the required subharmonic function is the *potential*

$$v_\mu(x) = \mu * h = \int_G h(x - y) d\mu_y. \tag{1.2.3}$$

Indeed, by § 1.5 the function $v_\mu(x)$ is subharmonic in R^m, and for any function $\varphi \in D(G)$

$$\begin{aligned}(\Delta v_\mu, \varphi) &= (v_\mu, \Delta\varphi) = \int_G \Delta\varphi d\omega_x \int_G h(x - y) d\mu_y \\ &= \int_G d\mu_y \int_G h(x - y)\Delta\varphi(x) d\omega_x = \int_G (h(x - y), \Delta\varphi(x)) d\mu_y \\ &= \int_G (\Delta_x h(x - y), \varphi(x)) d\mu_y = \theta_m \int_G (\delta(x - y), \varphi(x)) d\mu_y \\ &= \theta_m \int_G \varphi(y) d\mu_y.\end{aligned}$$

Consequently

$$\Delta v_\mu = \theta_m \mu. \tag{1.2.4}$$

REMARK. The essential point in the proof of (1.2.4) was that

$$\Delta_x h(x - y) = \delta(x - y).$$

Thus equality (1.2.4) continues to hold when the kernel $h(x - y)$ in (1.2.3) is replaced by any other kernel $h(x, y)$ such that $\Delta_x h(x, y) = \theta_m \delta(x - y)$.

When $\mu \equiv 0$, the only functions satisfying the condition $\Delta u = \theta_m \mu$, i.e. $\Delta u = 0$, are the harmonic functions. This assertion does not follow directly from the definition of a harmonic function, since the function $u(x)$ in the equation $\Delta u = 0$ is assumed a priori to be a distribution.

Thus, let the distribution $u(x)$ be such that $\Delta u = 0$. Consider the function

$$u_\varphi(x) = u * \varphi = (u(y), \varphi(x - y)), \tag{1.2.5}$$

where $\varphi \in D(E_\varepsilon(0))$. This function is defined on the open set

$$G_\varepsilon = \{x: E_\varepsilon(x) \Subset G\}$$

and is infinitely differentiable there (in the ordinary sense; see the Introduction, § 5.2). Moreover, by the rule for differentiation of a convolution,

$$\Delta u_\varphi = \Delta(u * \varphi) = \Delta u * \varphi = 0,$$

i.e. $u_\varphi(x)$ is harmonic on the set G_ε. Now consider the convolution $u_\varphi * \alpha_\varepsilon$, where $\alpha_\varepsilon(x)$ is the function defined in § 1.6. Repeating the argument given therein, we get

$$u_\varphi * \alpha_\varepsilon = \sigma_m \int_0^1 \alpha(s) s^{m-1} \mathfrak{M}_{u_\varphi}(x, s\varepsilon)\, ds.$$

Since $u_\varphi(x)$ is harmonic,

$$\mathfrak{M}_{u_\varphi}(x, s\varepsilon) = u_\varphi(x)$$

and hence, in the domain $G_{2\varepsilon}$,

$$u * \varphi * \alpha_\varepsilon = u_\varphi * \alpha_\varepsilon = u_\varphi = u * \varphi. \tag{1.2.6}$$

The convolution operation is in this case commutative and associative (see § 5.2 of the Introduction). It therefore follows from (1.2.6) that $u * \alpha_\varepsilon * \varphi = u * \varphi$, or what is the same,

$$(u * \alpha_\varepsilon, \varphi(x - y)) = (u(y), \varphi(x - y)). \tag{1.2.7}$$

Since φ is here any function in $D(E_\varepsilon(0))$, it follows that the equality

$$u * \alpha_\varepsilon = u \tag{1.2.8}$$

holds in every ball $E_\varepsilon(x)$, $x \in G_{2\varepsilon}$. The function $\alpha_\varepsilon(x)$ is in $D(E_{2\varepsilon}(0))$, and therefore the function $u * \varphi$, as a special case of the function u_φ considered above, is harmonic on the open set $G_{2\varepsilon}$. Because of (1.2.8) and the fact that $\varepsilon > 0$ is arbitrary, we see that $u(x)$ is a regular distribution, harmonic in the domain G.

We now have all the requisite preliminaries for the proof of the principal result in this section:

THEOREM 1.2.1. *Let the distribution $u(x) \in D^*(G)$ satisfy the condition $\Delta u \geqq 0$. Then it is a regular distribution, subharmonic in the domain G. Moreover, in each domain $G^0 \Subset G$ the function $u(x)$ admits the representation*

$$u(x) = v_{\mu_{G^0}}(x) + \Phi_{G^0}(x), \tag{1.2.9}$$

where $v_{\mu_{G^0}}(x) = h * \mu_{G^0}$, μ_{G^0} is the restriction of the measure $\mu = \Delta u/\theta_m$ to the domain G^0, and the function $\Phi_{G^0}(x)$ is harmonic in G^0.

For the proof, we consider the difference

$$\Phi_{G^0}(x) = u(x) - v_{\mu_{G^0}}(x).$$

It was shown above that $\Delta v_\mu = \theta_m \mu$ for any measure μ which is finite on G. In the domain G^0, therefore,

$$\Delta \Phi_{G^0}(x) = \Delta u - \Delta v_{\mu_{G^0}} = \Delta u - \theta_m \mu_{G^0} = \Delta u - \theta_m \mu = 0.$$

Consequently the distribution $\Phi_{G^0}(x)$ is regular and, moreover, harmonic in the domain G^0. This proves the representation (1.2.9), which in turn directly implies that the function $u(x)$ is subharmonic, since, as already remarked, the potential $v_{\mu_{G^0}}$ is subharmonic in the entire space R^m.

We now show that the representation (1.2.9) is unique. Indeed, if $u(x) = v_\nu(x) + \Phi(x)$ holds in the domain G^0, where ν is some measure which is finite on G^0 and $\Phi(x)$ a harmonic function in G^0, then

$$\Delta u = \Delta v_\nu + \Delta \Phi = \Delta v_\nu = \theta_m \nu,$$

i.e. $\nu = \mu_{G^0}$. Hence it follows immediately that $\Phi(x) = \Phi_{G^0}(x)$.

An immediate corollary of the result just proved and assertion (d) in § 1.4 is the well-known Riesz Decomposition Theorem, which antedates the theory of distributions:

THEOREM 1.2.1′ (F. RIESZ). *Let $u(x)$ be a function subharmonic in a domain $G \subset R^m$, $u(x) \not\equiv -\infty$. Then there is a unique measure μ in G such that in every subdomain $G^0 \Subset G$*

$$u(x) = \int_{G^0} h(x - y) d\mu_y + \Phi_{G^0}(x)$$

where $\Phi_{G^0}(x)$ is a harmonic function in G^0.

To end this subsection, we present a few properties of subharmonic functions which are very easy to prove using Theorem 1.2.1 (or, to be precise, the first statement in that theorem).[7] The operations applied below to derivatives of distributions are easily justified by approximating the arbitrary convex and subharmonic functions in question by suitable infinitely differentiable functions.

(a) *Any convex function $u(x)$ is subharmonic.*

Indeed, by the convexity of $u(x)$ we have $\partial^2 u / \partial x_i^2 \geqq 0$ for $i = 1, \cdots, m$, and consequently $\Delta u \geqq 0$. Hence, by Theorem 1.2.1, $u(x)$ is subharmonic.

(b) *Let $\varphi(t)$ be a convex, monotone nondecreasing function of $t \in R^1$, and $u(x)$ a subharmonic function in some domain $G \subset R^m$, such that the function $\varphi(u(x))$ is defined for all $x \in G$. Then the function $\Phi(x) = \varphi(u(x))$ is subharmonic.*

Indeed,

$$\Delta \Phi = \frac{\partial^2 \varphi}{\partial t^2} \sum_{i=1}^{m} \left(\frac{\partial u}{\partial x_i} \right)^2 + \frac{\partial \varphi}{\partial t} \Delta u,$$

and since by assumption

[7] These properties can also be proved without Theorem 1.2.1, but the proofs are then somewhat more difficult.

$$\partial^2\varphi/\partial t^2 \geqq 0, \qquad \partial\varphi/\partial t \geqq 0, \qquad \Delta u \geqq 0,$$

it follows that $\Delta\Phi \geqq 0$. Consequently $\Phi(x)$ is subharmonic.

(c) *If $u(x)$ is a subharmonic function, $e^{u(x)}$ is also subharmonic.*

This property, which is a special case of (b), may be given a slightly different formulation if we introduce the concept of logarithmically subharmonic function.

DEFINITION. A nonnegative function $u(x)$ is said to be *logarithmically subharmonic* if the function $\ln u(x)$ is subharmonic.

Proposition (c) may now be stated as follows:

Any logarithmically subharmonic function is subharmonic.

(d) *The sum of finitely many logarithmically subharmonic functions is logarithmically subharmonic.*

Indeed, let $u(x)$ and $v(x)$ be logarithmically subharmonic functions. Then $\Delta \ln u \geqq 0$ and $\Delta \ln v \geqq 0$. But an elementary computation shows that

$$\Delta \ln(u+v) = (u+v)^{-2}\Big\{u^2\Delta \ln u + v^2\Delta \ln v + uv\Delta \ln uv + uv\sum_{i=1}^{m}\left(\left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial v}{\partial x_i}\right)^2 - 2\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right)\Big\}.$$

Consequently $\Delta \ln (u+v) \geqq 0$, and thus $u+v$ is logarithmically subharmonic. The case of an arbitrary finite number of terms is obvious.

(e) *If the function $u(x, y)$ is continuous on a set $\{(x, y)\colon x \in G \subset R^m, y \in \bar{D} \subset R^n\}$, where G and D are domains in the indicated spaces, and the function $u(x, y)$ is logarithmically subharmonic in G as a function of x for each fixed $y \in \bar{D}$, then the function*

$$\Phi(x) = \int_D u(x, y)\, d\omega_y,$$

where $d\omega_y$ is the volume element in $R^n_{(y)}$, is also logarithmically subharmonic in G.[8]

This assertion follows in an obvious manner from the preceding property and the equality

$$\int_D u(x, y)d\omega_y = \lim \sum_{j=1}^{N} u(x, y^{(j)})\Delta\omega_j,$$

which defines $\Phi(x)$ as the limit of certain integral sums.

3. *Generalization of Jensen's formula.* Jensen's formula, well known in the theory of analytic functions of one variable, is

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta = \int_0^r \frac{n(t)}{t}\, dt + \ln|f(0)|. \tag{1.2.10}$$

[8] This assertion is also true under weaker conditions on the function $u(x,y)$.

Here $f(z)$ is a function analytic in the closed disk $\{z: |z| \leqq r\}$, and $n(t)$ is the number of its zeros in the disk $\{z: |z| \leqq t\}$. It is assumed, moreover, that $f(0) \neq 0$.

The following theorem gives a natural generalization of this formula to subharmonic functions.

THEOREM 1.2.2 (PRIVALOV [2]). *Let $u(x)$ be a function subharmonic in the ball $E_R(0)$, $u(x) \not\equiv -\infty$, μ the associated measure, $\mu(t) = \mu(\bar{E}_t(0))$, and*

$$\nu(t) = \frac{\theta_m}{\sigma_m} \frac{\mu(t)}{t^{m-2}}.$$

Then, if there is no mass in some neighborhood of the origin, i.e. $\mu(\varepsilon) = 0$ for some $\varepsilon > 0$, it follows that for any $r < R$

$$\mathfrak{M}_u(0, r) = \int_0^r \frac{\nu(t)}{t}\, dt + u(0).$$

PROOF. We first compute $\mathfrak{M}_{h_y}(0, r)$, where $h_y(x) = h(x - y)$. If $r < |y|$, then $h_y(x)$ is a harmonic function in the ball $E_r(0)$, and so

$$\mathfrak{M}_{h_y}(0, r) = h_y(0) = h(y). \tag{1.2.11}$$

To compute $\mathfrak{M}_{h_y}(0, r)$ for $r > |y|$, we use the elementary equality

$$|x - y| = (|y|/r)|x - y^*|, \tag{1.2.12}$$

which is valid for $|x| = r$, with y^* the image of the point y under inversion in the sphere $S_r(0)$. It follows from this equality that for $|x| = r$ the function

$$d_y(x) = h\left(\frac{|y|}{r}(x - y^*)\right)$$

satisfies the condition $d_y(x) = h_y(x)$. Consequently

$$\mathfrak{M}_{d_y}(0, r) = \mathfrak{M}_{h_y}(0, r). \tag{1.2.13}$$

On the other hand, since $d_y(x)$ is harmonic in the ball $E_{|y^*|}(0)$, it follows that for $|y| < r$

$$\mathfrak{M}_{d_y}(0, r) = d_y(0) = h_*(r).^{9)} \tag{1.2.14}$$

Equalities (1.2.13) and (1.2.14) imply that for $|y| < r$

$$\mathfrak{M}_{h_y}(0, r) = h_*(r). \tag{1.2.15}$$

Since the function $\mathfrak{M}_{h_y}(0, r)$ is monotone, it follows now from (1.2.15) and (1.2.11) that (1.2.15) is also valid for $r = |y|$.

By the Riesz Decomposition Theorem for the function $u(x)$ in the ball $E_r(0)$ we have

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9) For the definition of $h_*(r)$, see § 2.2.

$$u(x) = \int_{E_r(0)} h(x - y)d\mu_y + \Phi_r(x). \tag{1.2.16}$$

Averaging both sides of this equality over the sphere $S_{r'}(0)$, where $r' < r$, and using (1.2.11) and (1.2.15), we obtain

$$\begin{aligned}\mathfrak{M}_u(0, r') &= \int_{E_r(0)} \mathfrak{M}_{h_y}(0, r')d\mu_y + \mathfrak{M}_{\Phi_r}(0, r') = \int_{\bar{E}_{r'}(0)} + \int_{E_r\setminus\bar{E}_{r'}} \mathfrak{M}_{h_y}(0, r')d\mu_y + \Phi_r(0)\\ &= h_*(r') \int_{\bar{E}_{r'}(0)} d\mu + \int_{E_r\setminus\bar{E}_{r'}} h(y)d\mu_y + \Phi_r(0)\\ &= h_*(r')\mu(r') + \int_{E_r\setminus\bar{E}_{r'}} h(y)d\mu_y + \Phi_r(0).\end{aligned} \tag{1.2.17}$$

It is not difficult to see that

$$h_*(r' + 0)\mu(r' + 0) - h_*(r' - 0)\mu(r' - 0) = h_*(r') \int_{S_{r'}(0)} d\mu,$$

while

$$\int_{E_r\setminus\bar{E}_{r'+0}} - \int_{E_r\setminus\bar{E}_{r'-0}} h(y)d\mu_y = - h_*(r') \int_{S_r(0)} d\mu.$$

It therefore follows from (1.2.17) that the function $\mathfrak{M}_u(0, r')$ is continuous for any $r' < r$, and hence (since $r < R$ is arbitrary) for any $r' < R$. Using this and the fact that

$$\lim_{r'\to r-0} \mu(E_r \setminus \bar{E}_{r'}) = 0,$$

we obtain the following equality, by letting $r' \to r - 0$ in (1.2.17):

$$\mathfrak{M}_u(0, r) = h_*(r)\mu(r - 0) + \Phi_r(0). \tag{1.2.18}$$

The constant $\Phi_r(0)$ is found from (1.2.16):

$$\Phi_r(0) = u(0) - \int_{E_r(0)} h(y)d\mu_y.$$

Consider the integral in the right-hand side of this equality:

$$\begin{aligned}\int_{E_r(0)} h(y)d\mu_y &= \int_0^{r-0} h_*(t)d\mu(t) = h_*(r)\mu(r - 0) - \int_0^{r-0} \mu(t)dh_*(t)\\ &= h_*(r)\mu(r - 0) - \frac{\theta_m}{\sigma_m} \int_0^{r-0} \frac{\mu(t)}{t^{m-1}}\, dt = h_*(r)\mu(r - 0) - \int_0^r \frac{\nu(t)}{t}\, dt.\end{aligned}$$

Consequently

$$\Phi_r(0) = u(0) - h_*(r)\mu(r - 0) + \int_0^r \frac{\nu(t)}{t}\, dt.$$

Substituting this value of $\Phi_r(0)$ into (1.2.18), we now get

$$\mathfrak{M}_u(0, r) = \int_0^r \frac{\nu(t)}{t}\, dt + u(0).$$

This completes the proof of Theorem 1.2.2.

REMARK. If it is not assumed that there are no Riesz masses in some neighborhood of the origin, the following inequality is sometimes useful:

$$\frac{\theta_m}{\sigma_m} \int_\varepsilon^r \frac{\mu(t) - \mu(\varepsilon)}{t^{m-1}}\, dt + \mu(\varepsilon) h_*(r + \varepsilon) + C \geqq \mathfrak{M}_u(0, r)$$

$$\geqq \frac{\theta_m}{\sigma_m} \int_\varepsilon^r \frac{\mu(t) - \mu(\varepsilon)}{t^{m-1}}\, dt + \mu(\varepsilon) h_*(r - \varepsilon) + C, \tag{1.2.19}$$

where C is a constant independent of r.

To prove this inequality, it suffices to rewrite (1.2.16) in the form

$$u(x) = \int_{E_\varepsilon(0)} + \int_{E_r \setminus E_\varepsilon} h(x - y) d\mu_y + \Phi_r(x)$$

and to apply Jensen's formula to the sum of the last two terms (the first is estimated directly).

§ 3. Functions subharmonic in the entire space

In this section we consider the integral representation of functions which are subharmonic in R^m and the relation of their growth to their associated measures. The guiding point here is the analog with the representation of an entire function $f(z)$, $z \in C^1$, as a Weierstrass product. As is well known, every entire function $f(z)$ with zeros $a_1, a_2, \cdots$ satisfying the conditions

$$\sum_{j=1}^{\infty} \frac{1}{|a_j|^\alpha} < \infty \qquad (\alpha > 0)$$

admits a representation

$$f(z) = e^{g(z)} \prod_{j=1}^{\infty} \left\{ \left(1 - \frac{z}{a_j}\right) \exp\left\{ \sum_{l=1}^{m} \frac{1}{l} \left(\frac{z}{a_l}\right)^l \right\} \right\},$$

where $g(z)$ is an entire function and $m \geqq \alpha - 1$ an integer. Therefore the subharmonic function $\ln |f(z)|$ admits the representation

$$\ln|f(z)| = \operatorname{Re} g(z) + \int_{C^1} \left\{ \ln\left|1 - \frac{z}{a}\right| + \operatorname{Re} \sum_{l=1}^{m} \frac{1}{l} \left(\frac{z}{a}\right)^l \right\} d\mu_a,$$

where μ is a measure concentrated at the points a_j, $\mu(a_j)$ being equal to the multiplicity of the zero of $f(z)$ at a_j. We shall derive a similar representation for arbitrary subharmonic functions. The idea underlying the methods used here is related to that by which one develops the classical Weierstrass factorization of an entire function $f(z)$ of one variable.

1. *Order and convergence exponent.* Let $u(x)$ be a subharmonic function in the space R^m. Set

$$M_u(r) = \sup_{x \in E_r(0)} u^+(x),$$

where, as usual, $u^+(x) = \max\{0, u(x)\}$. Henceforth, when referring to the growth of a subharmonic function $u(x)$ we shall always mean the growth of the function $M_u(r)$. In particular, the order ρ_u of the function $u(x)$ is defined to be the order of $M_u(r)$:

$$\rho_u = \rho(M_u(r)) = \overline{\lim_{r \to \infty}} \frac{\ln M_u(r)}{\ln r}.$$

The order ρ_u characterizes the growth of the function $u(x)$. The associated measure is characterized by the convergence exponent.

DEFINITION. The *convergence exponent of a measure* μ defined in R^m is the greatest lower bound γ_μ of the numbers α for which the integral

$$\int^{\infty} \frac{\nu(t)}{t^{\alpha+1}}\, dt \tag{1.3.1}$$

converges,[10] where, as before,

$$\nu(t) = \frac{\theta_m}{\sigma_m} \frac{\mu(t)}{t^{m-2}},$$

with $\mu(t) = \mu(\bar{E}_t(0))$. An equivalent definition is obtained by replacing the integral (1.3.1) by

$$\int^{\infty} \frac{d\mu(t)}{t^{m-2+\alpha}}, \tag{1.3.2}$$

since these integrals converge and diverge together. Indeed,

$$\int^{r} \frac{d\mu(t)}{t^{m-2+\alpha}} = \frac{\mu(r)}{r^{m-2+\alpha}} + (m - 1 + \alpha) \int^{r} \frac{\mu(t)dt}{t^{m-2+\alpha}}$$

$$= \frac{\sigma_m}{\theta_m} \frac{\nu(r)}{r^{\alpha}} + (m - 2 + \alpha) \frac{\sigma_m}{\theta_m} \int^{r} \frac{\nu(t)}{t^{\alpha+1}}\, dt.$$

It follows immediately from this relation that convergence of the integral (1.3.2) implies that of the integral (1.3.1). If we assume that the integral (1.3.1) is convergent, then, as was shown in the Introduction, the function $\nu(t)$ is of minimal type for order α, i.e. $\nu(r)r^{-\alpha} \to 0$ as $r \to \infty$. Thus the integral (1.3.2) will also converge.

The order of an arbitrary subharmonic function $u(x)$ and the convergence ex-

[10] It follows from the results of the Introduction, § 4, that the convergence exponent of a measure μ is the order of the corresponding function $\nu(t)$.

ponent of its associated measure μ satisfy the simple inequality

$$\gamma_\mu \leqq \rho_\mu. \tag{1.3.3}$$

Indeed, it follows from (1.2.19) that

$$\int_1^r \frac{\nu(t)}{t}\, dt \leqq \frac{\sigma_m}{\theta_m} M_u(r) + 2\mu(1)h_*(r+1) + \text{const.} \tag{1.3.4}$$

It was shown in the Introduction, § 4, that the function

$$F(t) = \int_0^t \frac{f(s)}{s}\, ds,$$

where $f(s)$ is either a monotone nondecreasing function or the product of a monotone nondecreasing function and a function $s^{-\alpha}$, $\alpha > 0$, increases no slower than the function $f(t)$. It therefore follows from (1.3.4) that the function $\nu(r)$ increases no faster than $M_u(r)$. In particular, the order of the function $\nu(r)$ is at most ρ_u, i.e. for any $\varepsilon > 0$ there exist constants C_1 and C_2 such that for all $t > 0$

$$\nu(t) \leqq C_1 t^{\rho_u+\varepsilon} + C_2.$$

It is clear from this inequality that the integral (1.3.1) converges for all $\alpha > \rho_u$. Consequently $\gamma_\mu \leqq \rho_\mu$.

It is natural to ask whether for an arbitrary measure μ with $\gamma_\mu < \infty$ there exists a function $u(x)$, subharmonic in R^m, whose associated measure is μ, such that $\gamma_\mu = \rho_u$.

When the measure μ is finite in R^m, the answer is simple. For it then follows from the preceding section that the required function is the potential

$$v_\mu = \int_{R^m} h(x-y)\, d\mu_y.$$

Here $\gamma_\mu = \rho_u = 0$. In the general case this potential does not always exist, since the defining integral is divergent. However, as we shall show later, the kernel $h(x-y)$ may be modified in such a way that the integral will converge for measures μ for which the functions $\mu(t)$ are increasing, but not too rapidly, and the function defined by this integral will be subharmonic and have associated measure μ.

2. *Canonical kernel.* Expand the function $h(x-y)$ in the ball $E_{|y|}(0)$ as a Taylor series in powers of the coordinates of x:

$$h(x-y) = h(y) + D_1(x,y) + D_2(x,y) + \cdots.$$

Here $D_n(x,y)$, $n = 1, 2, \cdots$, denote terms homogeneous in x (polynomials of degree n) in the Taylor series of the function $h_y(x) = h(y-x)$. Note that for $y \neq 0$ every function $D_n(x,y)$ is a harmonic function of x in R^m, in fact, a harmonic polynomial. Indeed, $D_n(x,y)$ is the coefficient of t^n in the expansion of the function $h(tx-y)$ in powers of t. Consequently

$$D_n(x, y) = \frac{1}{n!}\left\{\frac{\partial^n}{\partial t^n} h(tx - y)\right\}\Big|_{t=0}.$$

The function $h(tx - y)$ is harmonic in x for $|x| < |y|/t$. Since differentiation with respect to t commutes with differentiation with respect to x, it follows that for any t in the ball $|x| < |y|/t$ the function $(\partial^n/\partial t^n)\, h(tx - y)$ is also a harmonic function of the variables $x_1, \cdots, x_m$. In particular, the functions

$$\left\{\frac{\partial^n}{\partial t^n} h(tx - y)\right\}\Big|_{t=0}$$

are also harmonic functions of x in R^m, and hence so is $D_n(x, y)$.

We now specify the function which will replace the kernel $h(x - y)$ in the sequel.

DEFINITION. The *canonical kernel of order* q $(q = 0,1,2,\cdots)$ is the function

$$h_q(x, y) = h(x - y) - h(y) - D_1(x, y) - \cdots - D_q(x, y).$$

That the canonical kernel $h_q(x, y)$ can be used instead of the ordinary kernel $h(x - y)$ follows from the easily verified fact that for any $y \neq 0$

$$\Delta_x h_q(x, y) = \theta_m \delta(x - y).$$

Indeed, the functions $D_n(x, y)$ are harmonic and, as was shown in § 2.2,

$$\Delta_x h(x - y) = \theta_m \delta(x - y).$$

Hence for $y \neq 0$

$$\Delta_x h_q(x, y) = \Delta_x \left\{h(x - y) - \sum_{n=1}^{q} D_n(x, y)\right\} = \theta_m \delta(x - y).$$

In order to estimate the canonical kernel later, we shall need an explicit expression for the function $D_n(x, y)$, and, correspondingly, for the kernel $h_q(x, y)$. We first consider the case $m > 2$. Let ψ denote the angle between the radius vectors of the points x and y, and let $\upsilon = |x|/|y|$. Then

$$\begin{aligned} h(tx - y) &= -\frac{1}{|tx - y|^{m-2}} = -\frac{1}{|y|^{m-2}} \frac{1}{(1 - 2\upsilon t \cos\psi + t^2\upsilon^2)^{(m-2)/2}} \\ &= -|y|^{2-m}(1 - t\upsilon e^{i\psi})^{1-m/2}(1 - t\upsilon e^{-i\psi})^{1-m/2}. \end{aligned} \tag{1.3.5}$$

Denote

$$A_n^\alpha = \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)}{n!}.$$

Using the fact that for $|\upsilon| < 1$

$$(1 - \upsilon)^{-\alpha} = 1 + \sum_{n=1}^{\infty} A_n^\alpha \upsilon^n,$$

we get the following equality, valid for $|x| < |y|/t$:

$$h(tx - y) = -\frac{1}{|y|^{m-2}}\left(1 + \sum_{n=1}^{\infty} A_n^{m/2-1} t^n \upsilon^n e^{-in\phi}\right)\left(1 + \sum_{n=1}^{\infty} A_n^{m/2-1} t^n \upsilon^n e^{-in\phi}\right)$$

$$= -\frac{1}{|y|^{m-2}}\left\{1 + \sum_{n=1}^{\infty} t^n \upsilon^n \sum_{k+l=n} A_k^{m/2-1} A_l^{m/2-1} e^{i(k-l)\phi}\right\}$$

$$= h(y) - \frac{1}{|y|^{m-2}} \sum_{n=1}^{\infty} t^n \upsilon^n \sum_{k+l=n} A_k^{m/2-1} A_l^{m/2-1} \cos(k - l)\phi. \qquad (1.3.6)$$

It follows from this equality that

$$D_n(x, y) = -\frac{1}{|y|^{m-2}} \upsilon^n \sum_{k+l=n} A_k^{m/2-1} A_l^{m/2-1} \cos(k - l)\phi. \qquad (1.3.7)$$

We introduce the notation

$$B_n^m(\phi) = \sum_{k+l=n} A_k^{m/2-1} A_l^{m/2-1} \cos(k - l)\phi$$

and rewrite (1.3.7) as

$$D_n(x, y) = -B_n^m(\phi)\upsilon^n / |y|^{m-2}. \qquad (1.3.8)$$

Correspondingly, we get the following equality for the kernel $h_q(x, y)$:

$$h_q(x, y) = -\frac{1}{|y|^{m-2}}(1 - 2\upsilon\cos\phi + \upsilon^2)^{1-m/2}$$

$$- h(y) + \sum_{n=1}^{q} \frac{B_n^m(\phi)\upsilon^n}{|y|^{m-2}} \qquad (y \neq 0) \qquad (1.3.9)$$

and

$$h_q(x, y) = -\sum_{n=q+1}^{\infty} \frac{B_n^m(\phi)\upsilon^n}{|y|^{m-2}} \qquad (|x| < |y|). \qquad (1.3.10)$$

The above arguments are valid for $m > 2$. If $m = 2$, then $h(x) = \ln|x|$. In this case,

$$h(tx - y) = \frac{1}{2}\ln\{(1 - t\upsilon e^{i\phi})(1 - t\upsilon e^{-i\phi})|y|^2\}$$

$$= \ln|y| - \sum_{n=1}^{\infty} \frac{(t\upsilon e^{i\phi})^n + (t\upsilon e^{-i\phi})^n}{2n} = \ln|y| - \sum_{n=1}^{\infty} \frac{t^n \upsilon^n \cos n\phi}{n}$$

and hence

$$D_n(x, y) = -\frac{\upsilon^n \cos n\phi}{n}. \qquad (1.3.11)$$

The kernel $h_q(x, y)$ then assumes the form

$$h_q(x, y) = \frac{1}{2} \ln (1 - 2v \cos \phi + v^2) + \sum_{n=1}^{q} \frac{v^n \cos n\phi}{n}. \tag{1.3.12}$$

Here x and y are arbitrary, except that $y \neq 0$. But if $|x| < |y|$, then

$$h_q(x, y) = -\sum_{n=q+1}^{\infty} \frac{v^n \cos n\phi}{n}. \tag{1.3.13}$$

3. *Estimation of the canonical kernel.* We first consider the case $m > 2$. The functions $B_n^m(\phi)$ are estimated for arbitrary ϕ as follows:

$$\begin{aligned} |B_n^m(\phi)| &\leqq \sum_{k+l=n} A_k^{m/2-1} A_l^{m/2-1} = A_n^{m-2} = \frac{(m-2)(m-1)\cdots(m-3+n)}{n!} \\ &= \left(1 + \frac{m-3}{1}\right)\left(1 + \frac{m-3}{2}\right)\cdots\left(1 + \frac{m-3}{n}\right) \\ &\leqq \exp\left\{(m-3)\sum_{l=1}^{n} \frac{1}{l}\right\} \leqq \exp\left\{(m-3)\left(1 + \int_1^n \frac{ds}{s}\right)\right\} = e^{m-3} n^{m-3}. \end{aligned} \tag{1.3.14}$$

We now estimate the kernel $h_q(x, y)$ for $|x|/|y| = v \leqq \tau$, where τ is an arbitrary fixed number in the interval (0,1). It follows from (1.3.10) and (1.3.14) that for $v \leqq \tau$

$$|h_q(x, y)| \leqq |y|^{2-m} v^{q+1} \sum_{n=q+1}^{\infty} |B_n^m(\phi)| \tau^{n-q-1} \leqq e^{m-3} |y|^{2-m} v^{q+1} \sum_{n=q+1}^{\infty} \tau^{n-q-1} n^{m-3}.$$

We now set

$$C_1 = e^{m-3} \sum_{n=q+1}^{\infty} \tau^{n-q-1} n^{m-3},$$

and rewrite the above equality as follows:

$$|h_q(x, y)| \leqq C_1 |y|^{2-m} v^{q+1} \qquad (v \leqq \tau). \tag{1.3.15}$$

No similar estimate of $h_q(x, y)$ is possible for $v > \tau$, since $h_q(y, y) = -\infty$. In this case we estimate $h_q(x, y)$ from above, using (1.3.9). Dropping the first two terms on the right-hand side of (1.3.9), we get the following inequality for $v > \tau$:

$$\begin{aligned} h_q(x, y) \leqq \sum_{n=1}^{q} \frac{B_n^m(\phi) v^n}{|y|^{m-2}} &\leqq |y|^{2-m} e^{m-3} v^q \sum_{n=1}^{q} n^{m-3} v^{n-q} \\ &\leqq |y|^{2-m} e^{m-3} v^q \sum_{n=1}^{q} n^{m-3} \tau^{n-q}. \end{aligned}$$

Setting

$$C_2 = e^{m-3} \sum_{n=1}^{q} n^{m-3} \tau^{n-q},$$

we rewrite this inequality as

$$h_q(x, y) \leqq C_2|y|^{2-m}\upsilon^q \qquad (\tau \leqq \upsilon < \infty). \tag{1.3.16}$$

From inequalities (1.3.15) and (1.3.16), which are valid in the intervals $0 < \upsilon \leqq \tau$ and $\tau \leqq \upsilon < \infty$, respectively, we get an inequality valid for all υ:

$$h_q(x, y) \leqq C\frac{|y|^{2-m}\upsilon^{q+1}}{1+\upsilon}, \tag{1.3.17}$$

where $C = \max\{(1+\tau)C_1, (1+\tau^{-1})C_2\}$. Inequalities (1.3.15) and (1.3.16), and hence also (1.3.17), are also valid for $m = 2$ (of course, with different constants C, C_1 and C_2). The reader will easily provide the proofs, which are exactly the same as for $m > 2$. The only difference is that the first term in (1.3.12) is not dropped but estimated as follows:

$$\frac{1}{2}\ln(1 - 2\upsilon\cos\phi + \upsilon^2) \leqq \frac{1}{2}\ln(1 + 2\upsilon + \upsilon^2) \leqq \ln(1+\upsilon) \leqq \upsilon \leqq \frac{\upsilon^q}{\tau^{q-1}}.$$

We have thus proved that for $m = 2, 3, \cdots$ and $q = 0, 1, \cdots$ there exists a constant $C = C(m, q)$ such that for any x and y, $y \neq 0$,

$$h_q(x, y) \leqq C|y|^{2-m}\frac{\upsilon^{q+1}}{1+\upsilon}, \tag{1.3.18}$$

or, what is the same thing,

$$h_q(x, y) \leqq C\frac{|x|^{q+1}}{|y|^{m-2+q}(|x| + |y|)} \qquad (y \neq 0). \tag{1.3.19}$$

4. *Canonical potential.* Let the measure μ vanish in some neighborhood of the origin and have finite convergence exponent γ_μ. We define the *canonical potential* of μ to be the function

$$J_\mu(x) = \int_{R^m} h_q(x, y)\, d\mu_y, \tag{1.3.20}$$

where q is the smallest of the numbers k for which the integral

$$\int_0^\infty \frac{d\mu(t)}{t^{k+m-1}}$$

is convergent. We now show that the canonical potential is well defined, i.e. the integral figuring in (1.3.20) exists.[11] Indeed, by (1.3.19) and the definition of q,

$$\int_{R^m} h_q(x, y)\, d\mu_y \leqq C\int_{R^m}\frac{|x|^{q+1}}{|y|^{m+q-2}(|x| + |y|)}\, d\mu_y$$

$$= C|x|^{q+1}\int_0^\infty \frac{d\mu(t)}{t^{m+q-2}(|x| + t)} \leqq C|x|^{q+1}\int_0^\infty \frac{d\mu(t)}{t^{m-1+q}} < \infty.$$

[11] Recall that by convention an integral diverging to $-\infty$ "exists."

The canonical potential J_μ has the same properties as the ordinary potential v_μ, which is defined for a much more restricted set of measures μ. To be precise, the following proposition holds:

The canonical potential J_μ is a subharmonic function. The Riesz measure associated with this subharmonic function is μ.

To prove this proposition, it will suffice, according to Theorem 1.2.1 and the definition of the associated measure, to show that the function $J_\mu(x)$ is upper semicontinuous and that

$$\Delta J_\mu = \theta_m \mu. \tag{1.3.21}$$

We omit the proof of (1.3.21), since, in view of the relation

$$\Delta_x h_q(x, y) = \theta_m \delta(x - y) \tag{1.3.22}$$

obtained above, it is exactly the same as the proof of the equality $\Delta v_\mu = \theta_m \mu$ in § 2.2 (see the Remark therein). In this connection the fact that (1.3.22) does not hold for $y = 0$ is not important, since by assumption the support of the measure μ does not contain the origin and hence, for some $\varepsilon > 0$,

$$J_\mu(x) = \int_{|y|>\varepsilon} h_q(x, y)\, d\mu_y.$$

Since for any $\varepsilon > 0$ the function $h_q(x, y)$ is obviously upper semicontinuous in the domain $\{(x, y): |x| < \infty, |y| \geqq \varepsilon\}$, it also follows immediately from this equality that the potential $J(x)$ is upper semicontinuous. This completes the proof of the proposition.

5. *Estimation of the canonical potential.* The canonical potential $J_\mu(x)$ has an extremely simple estimate in terms of the function

$$K_q(r; \nu) = r^q \left\{ \int_0^r \frac{\nu(t)}{t^{q+1}}\, dt + r \int_r^\infty \frac{\nu(t)}{t^{q+2}}\, dt \right\}.$$

Indeed, it follows from the estimate (1.3.19) that

$$J_\mu(x) \leqq C|x|^{q+1} \int_{R^m} \frac{d\mu_y}{|y|^{m+q-2}(|x| + |y|)} = C|x|^{q+1} \int_0^\infty \frac{d\mu(t)}{t^{m+q-2}(|x| + t)}$$

$$\leqq C|x|^q \int_0^{|x|} \frac{d\mu(t)}{t^{m-2+q}} + C|x|^{q+1} \int_{|x|}^\infty \frac{d\mu(t)}{t^{m-1+q}}.$$

Integrating by parts, we see that

$$J_\mu(x) \leqq C|x|^{1-m}\mu(|x|) + C(m - 2 + q)|x|^q \int_0^{|x|} \frac{\mu(t)}{t^{m+q-1}} dt$$

$$-C|x|^{1-m}\mu(|x|) + C(m - 1 + q)|x|^{q+1} \int_{|x|}^\infty \frac{\mu(t)}{t^{m+q}} dt$$

$$\leqq C(m-1+q)K_q\left(|x|;\frac{\mu(t)}{t^{m-2}}\right) = C(m-1+q)\frac{\sigma_m}{\theta_m}K_q(|x|;\nu(t)).$$

We have thus shown that every canonical potential $J_\mu(x)$ satisfies the inequality

$$J_\mu(x) \leqq C'K_q(|x|;\nu) \tag{1.3.23}$$

throughout R^m, with a suitable constant $C' = C'(m, q) < \infty$.

6. *Integral representation of subharmonic functions of finite order.* The following theorem answers the question posed in § 3.1.

THEOREM 1.3.1 (BRELOT [1]). *Let μ be a measure with finite convergence exponent γ_μ. Then there exists at least one subharmonic function $u(x)$ with associated measure μ such that $\gamma_\mu = \rho_u$.*

PROOF. If the measure μ vanishes at the origin, the required function is simply the canonical potential of the measure. Clearly we need only check the inequality $\gamma_\mu \geqq \rho_u$, where $u = J_\mu$. It follows from (1.3.23) that the order of $J_\mu(x)$ is at most that of $K_q(|x|;\nu)$. The order of $K_q(|x|;\nu)$ is at most that of $\nu(t)$ (see the Introduction, § 4) which, as we mentioned above, is equal to the convergence exponent of the measure μ. Thus $\rho_{J_\mu} \leqq \gamma_\mu$ and therefore $\rho_{J_\mu} = \gamma_\mu$.

If the support of the measure contains the origin, the required function is

$$u_\varepsilon(x) = J_{\mu_\varepsilon'}(x) + v_{\mu_\varepsilon}(x),$$

where ε is an arbitrary positive number, μ_ε is the restriction of μ to the ball $|x| < \varepsilon$, and μ_ε' is the restriction of μ to the exterior of this ball.

Indeed, using the properties of the potentials $J_{\mu_\varepsilon'}$ and v_{μ_ε}, we get

$$\Delta u_\varepsilon = \Delta J_{\mu_\varepsilon'} + \Delta v_{\mu_\varepsilon} = \theta_m\mu_\varepsilon' + \theta_m\mu_\varepsilon = \theta_m\mu$$

and

$$\rho(u_\varepsilon) \leqq \rho(J_{\mu_\varepsilon'}) + \rho(v_{\mu_\varepsilon}) = \rho(J_{\mu_\varepsilon'}) = \gamma_{\mu_\varepsilon'} = \gamma_\mu.^{12)} \quad \text{Q.E.D.}$$

We now observe that *any subharmonic function $u(x) \not\equiv -\infty$ whose associated measure μ has finite convergence exponent admits a representation*

$$u(x) = F(x) + \Phi(x), \tag{1.3.24}$$

where $F(x)$ is the subharmonic function associated with μ by Theorem 1.3.1, *and $\Phi(x)$ is a function harmonic throughout R^m.*

Indeed, since $\Phi(x) = u(x) - F(x)$, it follows that, throughout R^m,

$$\Delta\Phi = \Delta u - \Delta F = \theta_m(\mu - \mu) = 0,$$

and thus $\Phi(x)$ is harmonic.

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12) For convenience, we write $\rho(u)$ for ρ_u.

If the function $\Phi(x)$ is known to be of finite order, then it is a polynomial. More precisely, we have the following analog of Hadamard's theorem on the representation of entire functions of one variable:

THEOREM 1.3.2 (BRELOT [1]). *Let $u(x) \not\equiv -\infty$ be a subharmonic function in R^m, of finite order ρ. Then it admits the representation*

$$u(x) = J_{\mu_\varepsilon'}(x) + v_{\mu_\varepsilon}(x) + \Phi(x), \tag{1.3.25}$$

where ε is an arbitrary positive number, μ_ε is the restriction of the associated measure μ to the ball $|x| < \varepsilon$, μ_ε' the restriction of μ to the domain $|x| \geqq \varepsilon$, and $\Phi(x)$ a harmonic polynomial of degree at most q, where q is the smallest integer for which the integral $\int^\infty v_u(t)t^{-q-2}\,dt$ converges.

The proof of this theorem is based on the following lemmas.

LEMMA 1.3.1. *Let $u(x)$ be a subharmonic function in R^m. Then the function $M_u(r)$ increases no faster*[13] *than the function $\mathfrak{M}_u + (0, r)$.*

PROOF. The function $u^+(x)$ is obviously subharmonic. Therefore

$$\mathfrak{M}_{u^+}(0, r) \geqq \mathfrak{N}_{u^+}(0, r) \tag{1.3.26}$$

On the other hand, since $u^+(x)$ is nonnegative for $|x| = r/2$,

$$\mathfrak{N}_{u^+}(0, r) = \frac{1}{\omega_m r^m} \int_{E_r(0)} u^+ d\omega \geqq \frac{1}{\omega_m r^m} \int_{E_{r/2}(x)} u^+ d\omega = \frac{1}{2^m}\,\mathfrak{N}_{u^+}\left(x, \frac{r}{2}\right) \geqq \frac{u^+(x)}{2^m}.$$

Combining this with (1.3.26), we conclude that

$$M_u(r/2) \leqq 2^m \mathfrak{M}_{u^+}(0, r),$$

and thus the function $M_u(r)$ increases no faster than $\mathfrak{M}_{u^+}(0, r)$. Q.E.D.

LEMMA 1.3.2. *If $u(x)$ is subharmonic in R^m and $u(x) = u_1(x) - u_2(x)$, where the functions $u_1(x)$ and $u_2(x)$ are also subharmonic, then $u(x)$ increases no faster than the function* $\max [u_1, u_2]$.

PROOF. Express $u_2(x)$ as

$$u_2(x) = u_2^+(x) - u_2^-(x),$$

where $u_2^- = (-u_2)^+$. Then

$$u(x) = u_1(x) - u_2^+(x) + u_2^-(x) \leqq u_1(x) + u_2^-(x).$$

[13] Concerning the meaning of the term "no faster," see the Introduction, § 4.

Consequently $u^+(x) \leqq u_1^+(x) + u_2^-(x)$ and

$$\mathfrak{M}_{u^+} \leqq \mathfrak{M}_{u_1^+} + \mathfrak{M}_{u_2^-}. \tag{1.3.27}$$

We now estimate the function $\mathfrak{M}_{u_2^-}(0, r)$. By (1.2.19),

$$\mathfrak{M}_{u_2}(0, r) = \mathfrak{M}_{u_2^+} - \mathfrak{M}_{u_2^-} \geqq \frac{\theta_m}{\sigma_m} \int_\varepsilon^r \frac{\mu_{u_2}(t) - \mu_{u_2}(\varepsilon)}{t^{m-1}} dt + \mu_{u_2}(\varepsilon) h_*(r - \varepsilon) + \text{const.}$$

It follows that

$$\mathfrak{M}_{u_2^-}(0, r) \leqq \mathfrak{M}_{u_2^+}(0, r) + \mu_{u_2}(\varepsilon) h_*(r - \varepsilon) + \text{const.} \tag{1.3.28}$$

Using the obvious inequality $\mathfrak{M}_{u^+}(0, r) \leqq M_u(r)$, we now infer from (1.3.28) and (1.3.27) that

$$\mathfrak{M}_{u^+}(0, r) \leqq M_{u_1}(r) + M_{u_2}(r) + \mu_{u_2}(\varepsilon) h_*(r - \varepsilon) + \text{const.}$$

Since the function $h_*(r)$ is of zero order, the proof may now be completed by referring to Lemma 1.3.1.

COROLLARY. *If the function $\Phi(x)$ is harmonic, then the functions $-\Phi(x)$ and $|\Phi(x)|$ increase no faster than $\Phi(x)$.*

LEMMA 1.3.3. *A harmonic function of finite order not exceeding some constant $\rho < \infty$ is a polynomial of degree at most ρ. A harmonic function which is of at most minimal type for some integral order ρ is a polynomial of degree at most $\rho - 1$.*[14)]

PROOF. Let the function $\Phi(x)$ be harmonic in R^m and of order $\rho_\Phi \leqq \rho$. We represent the function $\Phi(x)$ in the ball $E_R(0)$ by the Poisson integral:

$$\Phi(x) = \frac{R^2 - |x|^2}{\sigma_m R} \int_{S_R(0)} \frac{\Phi(y) d\sigma_y}{(R^2 - 2R|x|\cos\psi + |x|^2)}$$

$$= \frac{R^2 - |x|^2}{\sigma_m R^{m+1}} \int_{S_R(0)} \frac{\Phi(y) d\sigma_y}{(1 - 2\upsilon\cos\psi + \upsilon^2)^{m/2}}.$$

The quantities υ and ψ are defined here just as in § 3.2. By § 3.2 the following expansion is valid for $|\upsilon| < 1$:

$$\frac{1}{(1 - 2\upsilon\cos\psi + \upsilon^2)^{m/2}} = 1 + \sum_{n=1}^{\infty} B_n^{m+2}(\psi)\upsilon^n,$$

where each term $B_n^{m+2}(\psi)\upsilon^n$ is a homogeneous polynomial of degree n in the coordinates of x. Substituting this expression into the Poisson integral, we get a power series expansion of the function $\Phi(x)$ in the ball $E_R(0)$:

[14)] This is the classical Liouville Theorem for harmonic functions.

$$\Phi(x) = (R^2 - |x|^2)\Phi(0) + \sum_{n=1}^{\infty} \frac{(R^2 - |x|^2)v^n}{R^{m+1}\sigma_m} \int_{S_R(0)} B_n^{m+2}(\psi)\Phi(y)d\sigma_y.$$

By the foregoing, each of the integrals figuring in this expansion is a homogeneous polynomial of degree n. Thus the homogeneous terms of the Taylor series for $\Phi(x)$ have the form

$$\frac{1}{n!}\frac{\partial^n\Phi(tx)}{\partial t^n}\bigg|_{t=0} = -\frac{|x|^2}{R^{m+1}\sigma_m}\int_{S_R(0)} B_{n-2}^{m+2}(\psi)v^{n-2}\Phi(y)d\sigma_y$$

$$+ \frac{1}{R^{m-1}}\int_{S_R(0)} B_n^{m+2}(\psi)v^n\Phi(y)d\sigma_y. \tag{1.3.29}$$

These terms are estimated as follows:

$$\max_{|x|=r}\left|\frac{1}{n!}\frac{\partial^n\Phi(tx)}{\partial t^n}\bigg|_{t=0}\right| \leqq \text{const}\,\frac{r^2}{R^{m+1}}\cdot\frac{r^{n-2}}{R^{n-2}}R^{m-1}M_{|\Phi|}(R)$$

$$+ \text{const}\,\frac{1}{R^{m-1}}\cdot\frac{r^n}{R^n}R^{m-1}M_{|\Phi|}(R) = \text{const}\left(\frac{r}{R}\right)^n M_{|\Phi|}(R).$$

The left-hand side of this inequality is independent of R. Therefore, letting $R \to \infty$ and using the Corollary to Lemma 1.3.2, we see that

$$\frac{\partial^n\Phi(tx)}{\partial t^n}\bigg|_{t=0} \equiv 0,$$

if the number n is greater than the order of $\Phi(x)$ (in particular, if $n > \rho$) or if the function $M_\Phi(R)$ is known a priori to be of at most minimal type for order ρ. Consequently the function $\Phi(x)$ is a polynomial of degree at most ρ, and strictly smaller than ρ if the function $\Phi(x)$ is of at most minimal type for order ρ. This proves the lemma.

Proceeding now to the proof of Theorem 1.3.2, we observe first that, as follows from the inequality $\gamma_\mu < \rho_u$, the function $u(x)$ has finite convergence exponent γ_μ. Therefore, as we proved above, it admits the representation $u(x) = F(x) + \Phi(x)$, where $\Phi(x)$ is a harmonic function and $F(x)$ the function constructed according to Theorem 1.3.1 from the measure μ_u associated with $u(x)$. By construction, $F(x) = J_{\mu_\varepsilon'} + v_{\mu_\varepsilon}$. Thus the representation (1.3.25) is valid with some harmonic function $\Phi(x)$. We claim that $\Phi(x)$ is a polynomial. Indeed,

$$\Phi(x) = u(x) - J_{\mu_\varepsilon'} - v_{\mu_\varepsilon},$$

and by Lemma 1.3.2 the rate of growth of the function $\Phi(x)$ is at most that of the function $\max[u, J_{\mu_\varepsilon'}, v_{\mu_\varepsilon}]$. The function v_{μ_ε} is of zero order, while $J_{\mu_\varepsilon'}$, as shown above, increases no faster than the corresponding function $K_q(r; \nu_u)$. Now the order of $K_q(r; \nu_u)$ is most $\rho(\nu_u) \leqq q + 1$, and it has minimal type for $\rho(\nu_u) = q + 1$.[15]

[15] See the Introduction, § 4.

Finally, it follows from (1.2.19) that the function $\nu_u(r)$ increases no faster than $M_u(r)$. Therefore $\Phi(x)$ has order at most ρ_u, and for $\rho_u = q + 1$ it is of minimal type. Hence by Lemma 1.3.3 it follows that $\Phi(x)$ is a polynomial of degree at most q. Q.E.D.

§ 4. Capacity of sets and related concepts

This section presents a few results from classical potential theory, without proofs.[16]

1. *Capacity.* Let K be a compact set in R^m, $m = 2,3,\cdots$. Let $\mathfrak{S}_m(K)$ denote the set of measures μ such that supp $\mu \subset K$ and $\inf_{x\in K} v_\mu(x) \geqq -1$.

DEFINITION. The (Wiener) *capacity* of a compact set K is the least upper bound $\mathrm{Cap}_m K$ (or simply Cap K) of the total masses $\mu \in \mathfrak{S}_m(K)$:

$$\mathrm{Cap}_m K = \sup_{\mu\in\mathfrak{S}_m(K)} \mu(K).$$

EXAMPLE 1. The capacity of a point is zero.

EXAMPLE 2. Let $K = \{z: z \in C^1, |z| \leqq r < 1\}$. Then

$$\mathrm{Cap}_2 K = 1/\ln(1/r). \tag{1.4.1}$$

Indeed, the function

$$\Phi(z) = \max\left\{-1, \frac{\ln|z|}{\ln(1/r)}\right\} \tag{1.4.2}$$

is obviously subharmonic of zero order, and therefore, by Theorem 1.3.2, admits the representation

$$\Phi(z) = v_\mu(z) + \text{const.} \tag{1.4.3}$$

Since $\Phi(z)$ is harmonic outside the circle $|z| = r$, the associated measure μ has support in K, and therefore, as is easily seen,

$$v_\mu(z) = \mu(K)\ln|z| + o(1) \tag{1.4.4}$$

for $z \to \infty$. Comparing (1.4.2), (1.4.3) and (1.4.4), we see that

$$\mu(K) = 1/\ln(1/r).$$

Now assume that there exists a measure $\mu' \in \mathfrak{S}_m(K)$ such that $\mu'(K) > \mu(K)$. Then, as follows from the properties of a potential, the function

$$u(z) = \Phi(z) - \frac{\mu(K)}{\mu'(K)} v_{\mu'}(z)$$

is harmonic outside the disk K, including the point at infinity, where $u(\infty) = 0$. On the other hand, on the circle $|z| = r$ we have $u(z) \leqq -1 + \mu(K)/\mu'(K) < 0$,

[16] For more details, see, for example, Landkof [1], Brelot [2], or Nevanlinna [1].

which contradicts the maximum principle. Consequently there is no measure μ with the above properties, and so $\mathrm{Cap}_2 K = 1/\ln(1/r)$.

EXAMPLE 3. The capacity of a ball in R^m, $m > 2$, is defined by

$$\mathrm{Cap}_m E_r = r^{m-2}.$$

The proof is analogous to that of (1.4.1).

Examination of these examples shows that for $m > 2$ the capacity of a ball is a finite nonnegative number, while for $m = 2$ it may be infinite. This unpleasant situation is avoided if we replace the Wiener capacity in the case $m = 2$ by the logarithmic capacity, which is defined for a compact subset K of the unit disk by

$$\mathrm{Cap}_2 K = \exp\left\{-\left\{\sup_{\mu \in \mathfrak{S}_2(K)} \mu(K)\right\}^{-1}\right\},$$

and for a compact subset K of the disk $|z| < r, r > 1$, by

$$\mathrm{Cap}_2 K = r\,\mathrm{Cap}_2 K_r$$

where K_r is the compact set obtained from K by a similarity transformation about the origin with ratio of similitude $1/r$.

It can be shown that this definition is independent of the choice of r. With this definition, the capacity of any compact set is finite; in particular, the capacity of a disk is equal to its radius. Henceforth any reference to the capacity of a set ($\mathrm{Cap}_m K$ or simply $\mathrm{Cap}\ K$) will be to the Wiener capacity if $m > 2$ and the logarithmic capacity if $m = 2$.

We note the following properties of the capacity of compact sets.

(a) *Capacity is a monotone increasing set function:* $\mathrm{Cap}\ K_1 \leqq \mathrm{Cap}\ K_2$ *if* $K_1 \subset K_2$.

(b) *The capacity of a set is invariant under "motions."*

(c) *Capacity is a subadditive set function; i.e., if* $K = \bigcup_1^\infty K_j$, *then* $\mathrm{Cap}\ K \leqq \sum_1^\infty \mathrm{Cap}\ K_j$.

(d) *Capacity is a convex set function; i.e., for any compact sets* K_1 *and* K_2,

$$\mathrm{Cap}\ (K_1 \cup K_2) + \mathrm{Cap}\ (K_1 \cap K_2) \leqq \mathrm{Cap}\ K_1 + \mathrm{Cap}\ K_2.$$

(e) *Capacity is a right-continuous set function; i.e., for any decreasing sequence of compact sets* $\{K_j\}$,

$$\lim_{j\to\infty} \mathrm{Cap}\, K_j = \mathrm{Cap}\left\{\bigcap_{j=1}^{\infty} K_j\right\}.$$

Now let A be an arbitrary set in R^m. We define the inner capacity $\underline{\mathrm{Cap}}\ A$ of A as the least upper bound of the capacities of all compact subsets K of A:

$$\underline{\mathrm{Cap}}\ A = \sup_{K \subset A} \mathrm{Cap}\ K.$$

The outer capacity $\overline{\mathrm{Cap}}\ A$ of A is defined as the greatest lower bound of the inner capacities of open supersets G of A:

$$\overline{\operatorname{Cap}} A = \inf_{G \supset A} \underline{\operatorname{Cap}} G.$$

It is obvious that $\overline{\operatorname{Cap}} A \geqq \underline{\operatorname{Cap}} A$. The above properties (c), (d) and (e) are valid for the inner capacities of Borel sets K_j. For outer capacity, properties (c), (d) and (e) are also valid for arbitrary sets. Properties (a) and (b) are clearly valid for arbitrary sets, for both inner and outer capacity.

If the inner and outer capacities of a set coincide, i.e. if $\underline{\operatorname{Cap}} A = \overline{\operatorname{Cap}} A$, the set A is said to be *capacitable*, and the common value of the inner and outer capacities of A is called its *capacity*, denoted as before by $\operatorname{Cap} A$ ($\operatorname{Cap}_m A$):

$$\operatorname{Cap} A = \underline{\operatorname{Cap}} A = \overline{\operatorname{Cap}} A.$$

It is readily seen that closed and open sets are capacitable. A much stronger result is the following theorem:

THEOREM 1.4.1 (CHOQUET [1]). *Any Borel set is capacitable.*[17]

The capacities of Borel sets possess properties (a) — (e) enumerated above for compact sets.

To end this subsection, we mention another important property of the capacity of a compact set, of a somewhat different nature from properties (a) — (e).

(f) *Let K be a compact set and E_∞ the component of the complement of K containing the point at infinity. Then* $\operatorname{Cap} K = \operatorname{Cap} (\partial E_\infty)$, *i.e. the capacity of a compact set is equal to the capacity of its outer boundary.*

It follows, in particular, from this property that *the capacity of the boundary of a bounded open set is different from zero.*

2. *Sets of zero capacity. Generalized maximum principle.* Sets of zero capacity play an essential role in the theory of subharmonic functions. In many cases these sets may be disregarded, only sets of positive capacity being important. In the sequel we shall encounter this situation repeatedly; for the moment, we illustrate it by the following proposition.

GENERALIZED MAXIMUM PRINCIPLE. *Let $u(x)$ be a function subharmonic and bounded above in a domain $G \subset R^m$. Assume that at all boundary points of G, except perhaps for a set of zero capacity,*

$$\overline{\lim_{\substack{x \to x_0 \in \partial G \\ x \in G}}} u(x) \leqq M,$$

where M is some constant. Then the inequality $u(x) \leqq M$ holds throughout G, and equality can hold only if $u(x) \equiv M$.

We note the following properties of sets of zero capacity:

(a) *The union of a countable set of Borel sets of zero capacity is again a set of zero capacity.*

[17] Choquet's Theorem is presented here in a slightly weakened form.

(b) *Let* $E \subset R^m$ *be a Borel set such that* $\operatorname{Cap}_m E = 0$. *Then the Lebesgue measure of* E *in the space* R^m *is also zero.*

(c) *Let* $E \subset R^m$, $m \geqq 2$, *be a Borel set such that* $\operatorname{Cap}_m E = 0$. *Then the* $(m-1)$*-dimensional Lebesgue measure of the intersection of* E *with any* $(m-1)$*-dimensional plane is zero.*

(d) *Any set in* R^m, $m > 2$, *which is contained in some* $(m-2)$*-dimensional plane has zero capacity in the space* R^m.

(e) *Let* $E \subset R^m$ *be a Borel set such that* $\operatorname{Cap} E = 0$. *Let the measure* μ *have support in* E *and* $\mu(E) \neq 0$. *Then the potential* $v_\mu(x)$ *cannot be bounded on* E.

3. *Robin distribution.* Consider the set of measures μ with support in a compact set $K \subset R^m$, $\operatorname{Cap} K > 0$, normalized by the condition $\mu(K) = 1$. This set contains a unique measure μ^* whose potential $v_{\mu^*}(x)$ is constant throughout K, except perhaps for a subset of zero capacity. This measure μ^* is called the *equilibrium measure* or *Robin distribution* of the compact set K, and the corresponding constant value of the potential $v_{\mu^*}(x)$, with its sign inverted, is called the *Robin constant* of K. This constant is usually denoted by $\gamma(K)$. Note that throughout the exterior of the compact set K we have $v_{\mu^*}(x) > -\gamma(K)$. When $m > 2$ the Robin constant of K is related to its capacity as follows:

$$\gamma(K) = 1/\operatorname{Cap} K,$$

and when $m = 2$ the analogous relation is

$$\gamma(K) = \ln(1/\operatorname{Cap} K).$$

The following property of the Robin distribution is extremely useful.

If μ *is the Robin distribution of a compact set* $K \subset R^m$, *then* $\mu(E) = 0$ *for any subset* $E \subset K$ *of zero capacity.*

4. *Thin sets.* A set $E \subset R^m$ is said to be *thin at a point* $x_0 \notin E$ if either $x_0 \notin \bar{E}$ or $x_0 \in \bar{E}$, but there exists a function $u(x)$, subharmonic in a neighborhood of x_0, such that

$$\overline{\lim_{\substack{x \to x_0 \\ x \in E}}} u(x) < u(x_0).$$

A set E is said to be thin at a point $x_0 \in E$ if the set $E \setminus x_0$ is thin at x_0.

A set which is not thin at a point x_0 is said to be *thick* at x_0. Thus, a set E is thick at x_0 if $x_0 \in \overline{E \setminus x_0}$ and, for any function $u(x)$ subharmonic in a neighborhood of x_0,

$$\overline{\lim_{\substack{x \to x_0 \\ x \in E}}} u(x) = u(x_0).$$

Note that *for any closed set* E *the set of points of* E *at which it is thin has zero capacity.*

There are various criteria for thin and thick sets. We shall need only one of these:

If E is a plane set such that all circles of sufficiently small radius about x_0 have nonempty intrsection with E, then E is thick at x_0.

5. *The Dirichlet problem.* Let $f(x)$ be a function defined and continuous on the boundary ∂G of a bounded domain $G \subset R^m$. Then there exists a unique function $\Phi_f(x)$ which is harmonic in G and satisfies the following conditions:

(a) The inequality $\Phi_f(x) \geqq u(x)$ $\forall x \in G$ holds for any function $u(x)$ subharmonic in G such that

$$\overline{\lim_{\substack{x \to x_0 \\ x \in G}}} u(x) \leqq f(x_0) \quad \forall x_0 \in \partial G.$$

(b) The inequality $\Phi_f(x) \leqq -u(x)$ $\forall x \in \partial G$ holds for any function $u(x)$ subharmonic in G such that

$$\overline{\lim_{\substack{x \to x_0 \\ x \in G}}} u(x) \leqq -f(x_0) \ \forall x_0 \in \partial G.$$

The function $\Phi_f(x)$ is called *the generalized solution of the Dirichlet problem for* $f(x)$. It is readily seen that if there exists an ordinary solution of the Dirichlet problem, i.e. a function $\Psi_f(x)$ harmonic in G, continuous in $\bar{G}$ and equal to $f(x)$ on the boundary of G, then the generalized solution and the ordinary solution coincide: $\Phi_f(x) = \Psi_f(x)$ for all $x \in G$.

The boundary points of G may be classified according to the behavior of the function $\Phi_f(x)$ in their neighborhood.

DEFINITION. A boundary point x_0 of a bounded domain G is said to be *regular* if the condition

$$\lim_{\substack{x \to x_0 \\ x \in G}} \Phi_f(x) = f(x_0)$$

holds for any function $f(x)$ continuous on ∂G. Boundary points of G which are not regular are said to be *irregular*.

In the terminology of the preceding subsection, the boundary points may be characterized as follows.

A point x_0 on the boundary ∂G of a bounded domain $G \subset R^m$ is regular if and only if the set $R^m \setminus G$ is not thin at x_0.

This proposition implies the following corollaries:

(a) *The property of a point to be regular or irregular is local, i.e. it depends on the properties of the set ∂G in a neighborhood of the point.*

(b) *The set of irregular points of the boundary has zero capacity, and therefore the function $\Phi_f(x)$ can be defined at the points $x \in \partial G$ by $\Phi_f(x) = f(x)$, and the resulting function is continuous throughout $\bar{G}$ except perhaps for some set of zero capacity on ∂G.*

6. *Harmonic measure.* Let K be a compact set in the ball $E_r(x_0) \subset R^m$. Let $G_r(K)$ denote the component of the open set $E_r(x_0) \setminus K$ whose boundary contains the

sphere $S_r(x_0)$. We let $\partial_\infty K$ denote the outer boundary of the compact set K, i.e. the set of points of K which are boundary points of $G_r(K)$.

DEFINITION. The *harmonic measure of the compact set* K relative to the ball $E_r(x_0) \supset K$ is the function $\omega_r(x) = \omega_r(x, K, x_0)$, harmonic in the domain $G_r(K)$, whose limiting values

$$\lim_{\substack{x \to x' \in \partial G_r(K) \\ x \in G_r(K)}} \omega_r(x)$$

vanish throughout the sphere $S_r(x_0)$ and equal unity at all points of the set $\partial_\infty K$, except perhaps for a set of zero capacity.

The existence of the function $\omega_r(x)$ follows from the properties of the generalized solution of the Dirichlet problem, considered in the preceding subsection.

Harmonic measure is often employed in estimates of subharmonic functions. Thus, if the function $u(x)$ is subharmonic in the domain $G_r(K)$ and

$$\overline{\lim_{\substack{x \to x' \\ x \in G_r(K)}}} u(x) \leqq M \qquad \forall x' \in S_r(x_0),$$

$$\overline{\lim_{\substack{x \to x' \\ x \in G_r(K)}}} u(x) \leqq m \qquad \forall x' \in \partial_\infty K,$$

where M and m are constants, then, by the generalized maximum principle, the following inequality holds throughout $G_r(K)$:

$$u(x) \leqq m\omega_r(x) + M(1 - \omega_r(x)). \tag{1.4.5}$$

Later we shall make use of the following estimate for the harmonic measure:

LEMMA 1.4.1. *Let K be a plane closed subset of the disk $\{z: z \in C_1, |z| < r_0\}$, with positive capacity. Then, for any $r > 2r_0$, the harmonic measure $\omega_r(z)$ of K relative to the disk $|z| < r$ satisfies the following inequality in the annulus $r_0 < |z| < r$:*

$$\omega_r(z) \geqq 1 - \frac{\ln(|z| + r_0) - \ln \operatorname{Cap} K}{\ln(r - r_0) - \ln r_0}.$$

PROOF. Let $v_\mu(x)$ be the logarithmic potential of the equilibrium measure μ of K. Denote

$$m(t) = \min_{|z| = t} v_\mu(z)$$

and consider the function

$$1 - \omega_r(z) - \frac{v_\mu(z) + \gamma}{m(r) + \gamma},$$

where $\gamma = \gamma(K)$ is the Robin constant of K. It is easily seen that this function is bounded and harmonic in the domain $G_r(K)$ and has nonpositive boundary

values on all of ∂G_r, except perhaps for a set of zero capacity. Throughout $G_r(K)$, therefore, we have the inequality

$$1 - \omega_r(z) - \frac{\upsilon_\mu(z) + \gamma}{m(r) + \gamma} \leqq 0,$$

whence

$$\omega_r(r) \geqq 1 - \frac{\upsilon_r(z) + \gamma}{m(r) + \gamma}.$$

The estimate of $\upsilon_\mu(z)$ for $|z| > r_0$ is obvious:

$$\upsilon_\mu(z) = \int_K \ln|z - \zeta| \, d\mu_\zeta \leqq \mu(K) \ln(|z| + r_0) = \ln(|z| + r_0).$$

Similarly, $\upsilon_\mu(z) \geqq \ln(|z| - r_0)$. Therefore $m(r) \geqq \ln(r - r_0)$ and

$$\omega_r(z) \geqq 1 - \frac{\ln(|z| + r_0) + \gamma}{\ln(r - r_0) + \gamma}$$

$$= 1 - \frac{\ln(|z| + r_0) - \ln \operatorname{Cap} K}{\ln(r - r_0) - \ln \operatorname{Cap} K} \geqq 1 - \frac{\ln(|z| + r_0) - \ln \operatorname{Cap} K}{\ln(r - r_0) - \ln r_0}.$$

Q.E.D.

§ 5. Regularization

1. *Definition and simplest properties of regularization.* The concept of regularization to be studied here plays a major role in our subsequent constructions.

DEFINITION. The (upper) *regularization* of a real-valued function $u(x)$ is the function

$$u^*(x) = \lim_{\varepsilon \to 0} \sup_{x' \in E_\varepsilon(x)} u(x'). \tag{1.5.1}$$

The regularization $u^*(x)$ may also be defined as the *smallest upper semicontinuous majorant* of the function $u(x)$. Indeed, by (1.5.1) we have $u^*(x) \geqq u(x)$, and for any number $M > u^*(x)$ there exists a number $\delta = \delta(\varepsilon, x) > 0$ such that

$$\sup_{x' \in E_\delta(x)} u(x') \leqq M.$$

Consequently, for $x' \in E_\delta(x)$,

$$u^*(x') = \lim_{\varepsilon \to 0} \sup_{x \in E_\delta(x')} u(x'') \leqq M$$

and thus the function $u^*(x)$ is upper semicontinuous at x. Now any upper semicontinuous function $u_1(x)$ which satisfies the inequality $u_1(x) \geqq u(x)$ for all x obviously satisfies the inequality $u_1^*(x) \geqq u^*(x)$. It follows immediately from the definition of upper semicontinuity that $u_1^*(x) \leqq u_1(x)$, so that $u_1^*(x) = u_1(x)$, and hence $u_1(x) \geqq u^*(x)$.

It was mentioned in § 1 that the least upper bound of any finite family of subharmonic functions is subharmonic. This is no longer true for infinite families of subharmonic functions, since in that case semicontinuity may be lost. However, a related result, involving regularization, is valid:

Let $\mathfrak{F} = \{u_\alpha(x)\}$ *be a family of subharmonic functions, uniformly bounded above in the domain* $D \subset R^m$, *where* α *ranges over an arbitrary index set. Let*

$$u(x) = u_{\mathfrak{F}}(x) = \sup_{\mathfrak{F}} u_\alpha(x).$$

Then the regularization $u^*(x)$ *of the function* $u(x)$ *is subharmonic in* D.

Indeed, the inequality

$$u_\alpha(x) \leqq \mathfrak{M}_{u_\alpha}(x, r), \tag{1.5.2}$$

valid for $E_r(x) \subset D$ for any function $u_\alpha(x)$, implies that

$$u(x) \leqq \mathfrak{M}_u(x, r). \tag{1.5.3}$$

Similarly, for $E_{r+\delta}(x) \subset D$ we have

$$\sup_{x' \in E_\delta(0)} u(x + x') \leqq \frac{1}{\sigma_m r^{m-1}} \int_{S_r(0)} \left\{ \sup_{x' \in E_\delta(0)} u(x + x' + x'') \right\} d\sigma_{x''}.$$

Letting $\delta \to 0$ in this inequality, we see that $u^*(x) \leqq \mathfrak{M}_{u^*}(x, r)$. Combining this inequality with the fact that the regularization is upper semicontinuous, we see that $u^*(x)$ is subharmonic.

If the index set of the family $\mathfrak{F}$ contains real numbers α arbitrarily close to $+\infty$, we consider, besides the function $u(x)$ defined above, also the function

$$v(x) = v_{\mathfrak{F}}(x) = \overline{\lim_{\alpha \to \infty}} u_\alpha(x).$$

We claim that the regularization $v^*(x)$ of $v(x)$ is again a subharmonic function. Indeed, it follows from (1.5.2) that

$$\sup_{\alpha > N} u_\alpha(x) \leqq \frac{1}{\sigma_m r^{m-1}} \int_{S_r(x)} \left\{ \sup_{\alpha > N} u_\alpha(x') \right\} d\sigma_{x'}.$$

Letting $N \to \infty$, we get an inequality analogous to (1.5.3):

$$v(x) \leqq \mathfrak{M}_v(x, r),$$

and the subharmonicity of $v^*(x)$ is now proved in the same way as that of $u^*(x)$.

The functions $u(x)$ and $v(x)$ defined above in terms of the family $\mathfrak{F}$ are related to their regularizations, the subharmonic functions $u^*(x)$ and $v^*(x)$, by the obvious inequalities $u(x) \leqq u^*(x)$ and $v(x) \leqq v^*(x)$. Let $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$ denote the sets of points at which these inequalities are strict. In other words,

$$T_{\mathfrak{F}} = \{x : u(x) < u^*(x)\}, \qquad S_{\mathfrak{F}} = \{x : v(x) < v^*(x)\}.$$

These sets play an important part in applications of subharmonic functions to the

theory of analytic functions. Later in this section we shall show that the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$ are in a certain sense "lean," and hence the least upper bound $u(x)$ of the family $\mathfrak{F}$ of subharmonic functions, and its upper limit $v(x)$, though not subharmonic functions, nevertheless coincide with subharmonic functions except perhaps for certain exceptional sets.

2. *The structure of the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$.*[18] We first consider the case of a countable family $\mathfrak{F}$.

LEMMA 1.5.1. *Let the function $\Phi(x)$ be upper semicontinuous in a domain D. Let*

$$T_{\mathfrak{F}}(\Phi) = \{x: u(x) \leqq \Phi(x)\}.$$

Then, if the family $\mathfrak{F}$ is countable, the set $T_{\mathfrak{F}}(\Phi)$ is a G_δ set. If the functions $u_\alpha \in \mathfrak{F}$ are also continuous, $T_{\mathfrak{F}}(\Phi)$ is closed in D.

PROOF. By definition, $u(x) = \sup_\alpha u_\alpha(x)$. Therefore

$$T_{\mathfrak{F}}(\Phi) = \bigcap_\alpha \{x: u_\alpha(x) \leqq \Phi(x)\}. \tag{1.5.4}$$

We first show that the sets $\{x: u_\alpha(x) \leqq \Phi(x)\}$ figuring in this relation are G_δ sets. Indeed, since $\Phi(x)$ is upper semicontinuous, we have $\Phi(x) = \lim \Phi_j(x)$, where the functions $\Phi_j(x)$ are continuous in the closures $\bar{D}_j$ of certain domains D_j ($D_j \subset D_{j+1}$, $\bigcup D_j = D$), and $\Phi_j(x) \geqq \Phi_{j+1}(x)$ $\forall x \in \bar{D}_j$, $j = 1,2,\cdots$. Then, as is readily seen,

$$\{x: u_\alpha(x) \leqq \Phi(x)\} = \left\{x: u_\alpha(x) \leqq \lim_{j\to\infty} \Phi_j(x)\right\}$$

$$= \bigcap_{j=1}^{\infty} \left\{\left\{x: x \in \bar{D}^j, u_\alpha(x) < \Phi_j(x) + \frac{1}{j}\right\} \cap \{D \setminus \bar{D}_j\}\right\}.$$

Since the functions $\Phi_j(x)$ are continuous and the functions $u_\alpha(x)$ semicontinuous, any set

$$\left\{x: x \in \bar{D}_j,\ u_\alpha(x) < \Phi_j(x) + \frac{1}{j}\right\} \cup \{D \setminus \bar{D}_j\}$$

is open. Consequently $\{x: u_\alpha(x) \leqq \Phi(x)\}$ is indeed a G_δ set. Now the intersection of countably many G_δ sets is again a G_δ set, and hence this is true of the set $T_{\mathfrak{F}}(\Phi)$ (see (1.5.4)). If the functions $u_\alpha(x)$ are also continuous, each set $\{x: u_\alpha(x) \leqq \Phi(x)\}$ is obviously closed in D and therefore so is $T_{\mathfrak{F}}(\Phi)$. Q.E.D.

LEMMA 1.5.2. *If the family $\mathfrak{F}$ is countable, the set $T_{\mathfrak{F}}$ is a $G_{\delta\sigma}$ set. If the functions in $\mathfrak{F}$ are continuous, $T_{\mathfrak{F}}$ is an F_σ set.*

[18] The family $\mathfrak{F}$ is the same as in § 1.5.1.

PROOF. The statement of the lemma follows immediately from Lemma 1.5.1 and the obvious equality

$$T_{\mathfrak{F}} = \bigcup_{j=1}^{\infty} T_{\mathfrak{F}}(u^* - \varepsilon_j),$$

where $\varepsilon_j \to 0$ as $j \to \infty$, and the set $T_{\mathfrak{F}}(u^* - \varepsilon_j)$ is the set $T_{\mathfrak{F}}(\Phi)$ of Lemma 1.5.1 with $\Phi = u^* - \varepsilon_j$.

LEMMA 1.5.3. *Let the function $\Phi(x)$ be upper semicontinuous in the domain D. Let*

$$S_{\mathfrak{F}}(\Phi) = \{x: v(x) < \Phi(x)\}.$$

Then, if the family $\mathfrak{F}$ is countable, $S_{\mathfrak{F}}(\Phi)$ is a $G_{\delta\sigma}$ set. If the functions $u_\alpha(x) \in \mathfrak{F}$ are continuous, $S_{\mathfrak{F}}(\Phi)$ is an F_σ set.

PROOF. It follows from the equality

$$v(x) = \lim_{n\to\infty} \sup_{\alpha \geqq n} u_\alpha(x)$$

that the set $S_{\mathfrak{F}}(\Phi)$ admits the representation

$$S_{\mathfrak{F}}(\Phi) = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{x: \sup_{\alpha \geqq n} u_\alpha(x) \leqq \Phi(x) - \frac{1}{k}\right\}. \tag{1.5.5}$$

By Lemma 1.5.1 the set

$$\left\{x: \sup_{\alpha \geqq n} u_\alpha(x) \leqq \Phi(x) - \frac{1}{k}\right\}$$

is a G_δ set, and if the functions $u_\alpha(x)$ are continuous it is closed in the domain D. From this and from (1.5.5) it follows immediately that in the general case $S_{\mathfrak{F}}(\Phi)$ is a $G_{\delta\sigma}$ set, and when the functions $u_\alpha(x)$ are continuous it is an F_σ set in the domain D. Q.E.D.

LEMMA 1.5.4. *If the family $\mathfrak{F}$ is countable, then the set $S_{\mathfrak{F}}$ of points $x \in D$ at which the inequality $v(x) < v^*(x) - \varepsilon$ holds for given $\varepsilon \geqq 0$ is a $G_{\delta\sigma}$ set. If the functions in $\mathfrak{F}$ are continuous, $S_{\mathfrak{F}}$ is an F_σ set.*

PROOF. The lemma follows immediately from Lemma 1.5.3 with the function $\Phi(x)$ replaced by $v^*(x) - \varepsilon$.

REMARK. Since $S_0 = S_{\mathfrak{F}}$, it follows in particular from Lemma 1.5.4 that for a countable family $\mathfrak{F}$ the set $S_{\mathfrak{F}}$ is always a $G_{\delta\sigma}$ set, and if the functions $u_\alpha(x)$ are continuous it is an F_σ set in D.

We now turn to the case of an arbitrary, not necessarily countable family $\mathfrak{F}$. The following lemma is then very useful.

LEMMA 1.5.5 (CHOQUET).[19] *Any family* $\mathfrak{F}$ *contains a countable subfamily* $\mathfrak{F}_0$ *such that the corresponding sets* $T_{\mathfrak{F}}$ *and* $T_{\mathfrak{F}_0}$ *satisfy the relation* $T_{\mathfrak{F}} \subset T_{\mathfrak{F}_0}$.

PROOF. Consider some sequence of balls $E_n = E_{\varepsilon_n}(x^{(n)})$ satisfying the following conditions:

(a) $E_n \subset D$, $n = 1,2,\cdots$.

(b) $\lim_{n\to\infty} \varepsilon_n = 0$.

(c) For every point $x' \in D$ there exists an infinite subsequence $\{E_{n_j}\}$ such that $x' \in E_{n_j}$ for every j.

That such a sequence exists is obvious. It is readily seen that for every ball E_n there exists a function $u_{\alpha_n}(x)$ satisfying the inequality

$$\sup_{E_n} u_{\alpha_n}(x) \geqq \sup_{E_n} u(x) - \frac{1}{n}.$$

Denote the set of all the functions $u_{\alpha_n}(x)$ by $\mathfrak{F}_0$, and let

$$u_0(x) = u_{\mathfrak{F}_0}(x) = \sup_{\mathfrak{F}_0} u_\alpha(x).$$

Obviously,

$$u_0(x) \leqq u(x), \tag{1.5.6}$$

and consequently

$$u_0^*(x) \leqq u^*(x). \tag{1.5.7}$$

On the other hand, $u_0^*(x) \geqq u^*(x)$ for all $x \in D$. Indeed, let $x' \in D$ and consider a subsequence of indices n_j such that $x' \in E_{n_j}$, $j = 1,2,\cdots$. Then

$$\sup_{E_{n_j}} u_0(x) \geqq \sup_{E_{n_j}} u_{\alpha_{n_j}}(x) \geqq \sup_{E_{n_j}} u(x) - \frac{1}{n_j}.$$

Letting $j \to \infty$ in this inequality, we get

$$u_0^*(x') \geqq u^*(x') \qquad \forall x' \in D. \tag{1.5.8}$$

It follows from (1.5.7) and (1.5.8) that

$$T_{\mathfrak{F}} = \{x: u(x) < u^*(x)\} \subset \{x: u_0(x) < u^*(x)\} = \{x: u_0(x) < u_0^*(x)\} = T_{\mathfrak{F}_0}.$$

From this and inequality (1.5.6) we conclude that $u_0^*(x') = u^*(x')$ for all $x' \in D$. Thus the countable family $\mathfrak{F}_0$ is such that $T_{\mathfrak{F}} \subset T_{\mathfrak{F}_0}$. Q.E.D.

If the functions $u_\alpha(x)$ of the family $\mathfrak{F}$ are continuously dependent on the parameter α, we have a stronger assertion than Lemma 1.5.5:

LEMMA 1.5.6. *Let the family* $\mathfrak{F} = \{u_\alpha(x)\}$ *be indexed by the points* α *in an entire interval* $I = (a, \infty)$, *and assume that for fixed* x *the functions* $u_\alpha(x)$ *are continuous*

[19] See, for example, Brelot [2].

in the parameter α. Then there exists a countable subfamily $\mathfrak{F}_0$ of $\mathfrak{F}$ such that the corresponding sets $T_{\mathfrak{F}_0}$ and $S_{\mathfrak{F}_0}$ satisfy the conditions $T_{\mathfrak{F}_0} = T_{\mathfrak{F}}$ and $S_{\mathfrak{F}_0} = S_{\mathfrak{F}}$.

PROOF. The required subfamily is any family $\mathfrak{F}_0 = \{u_\alpha\}_{\alpha \in I_0}$, where I_0 is an arbitrary dense subset of I. Indeed, by the continuity of the function $u_\alpha(x)$ in α we have

$$u(x) = \sup_{\alpha \in I} u_\alpha(x) = \sup_{\alpha \in I_0} u_\alpha(x),$$

$$v(x) = \overline{\lim_{\alpha \to \infty}} u_\alpha(x) = \overline{\lim_{\substack{\alpha \to \infty \\ \alpha \in I_0}}} u_\alpha(x).$$

Consequently $T_{\mathfrak{F}} = T_{\mathfrak{F}_0}$ and $S_{\mathfrak{F}} = S_{\mathfrak{F}_0}$. Q.E.D.

REMARK. In the proofs of Lemmas 1.5.1 to 1.5.4 we used only the semicontinuity of the functions $u_\alpha(x)$, not their subharmonicity; the proofs of Lemmas 1.5.5 and 1.5.6 did not use even this weak condition. Lemmas 1.5.1 to 1.5.6 are therefore valid for less restricted families of functions than those considered above; *in particular, they are valid for families of upper semicontinuous functions.*

3. *Capacity of the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$.* Our first step in estimating the "richness" of the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$ is to prove the following lemma.

LEMMA 1.5.7. *As before, let $u(x) = \sup_{\mathfrak{F}} u_\alpha(x)$, where $\mathfrak{F} = \{u_\alpha(x)\}$ is a family of functions subharmonic in a domain $G \subset R^m$. Let $\Phi(x)$ be a subharmonic function in G such that $u(x) \leqq \Phi(x)$ for all $x \in G$, and assume that, on some compact subset $K \subset G$ of positive capacity,*

$$u(x) \leqq \Phi(x) - \varepsilon. \tag{1.5.9}$$

Then there is a point x_0 in G such that $u^*(x_0) < \Phi(x_0) - \varepsilon/5$ (the number $\varepsilon/5$ may be replaced by any positive number $\varepsilon_1 < \varepsilon$).

PROOF. Since K is compact, its boundary is contained in the boundary of one of the components of the complement of K in G. Denote this component by D. We saw above that the set of irregular points on the boundary of any domain has capacity zero. On the other hand, by (e) in § 4.1 the capacity of the outer boundary of a compact set is equal to the capacity of the entire compact set, and is therefore positive. Hence there are regular points on the boundary of D which are not points of ∂G. Let x_0 be one of these. Since the function $\Phi(x)$ is upper semicontinuous, for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon)$ such that the inequality

$$\Phi(x) \leqq \Phi(x_0) + \varepsilon/3 \tag{1.5.10}$$

holds in the ball $E_{2\delta}(x_0)$. Set $K' = K \cap \bar{E}_\delta(x_0)$. The complement of K' in the ball $E_{2\delta}(x_0)$ is an open set, and therefore the union of at most countably many components $D_0, D_1, \cdots$. To fix ideas, let D_0 be the component whose boundary contains the sphere $S_{2\delta}(x_0)$. It is clear that $x_0 \in \bar{D}_0$. Moreover, since regularity is a local prop-

erty, as mentioned above, x_0 is also a regular point of the bounary of D_0. Consequently at this point the harmonic measure $\omega_{2\delta}(x) = \omega_{2\delta}(x, K', x_0)$ of the compact set K' relative to the ball $E_{2\delta}(x_0)$ satisfies the relation

$$\lim_{\substack{x \to x_0 \\ x \in D_0}} \omega_{2\delta}(x) = 1. \tag{1.5.11}$$

Let us estimate each of the functions $u_\alpha(x)$ in terms of $\omega_{2\delta}(x)$. According to (1.4.5), (1.5.9) and (1.5.10) we have

$$\begin{aligned} u_\alpha(x) &\leqq \omega_{2\delta}(x) \sup_{x \in K'} u_\alpha(x) + (1 - \omega_{2\delta}(x)) \sup_{x \in S_{2\delta}(x_0)} u_\alpha(x) \\ &\leqq \omega_{2\delta}(x) \sup_{x \in K'} (\Phi(x) - \varepsilon) + (1 - \omega_{2\delta}(x)) \sup_{x \in S_{2\delta}(x_0)} \Phi(x) \\ &\leqq \omega_{2\delta}(x)(\Phi(x_0) + \varepsilon/3 - \varepsilon) + (1 - \omega_{2\delta}(x))(\Phi(x_0) + \varepsilon/3) \\ &= \Phi(x_0) + \varepsilon/3 - \varepsilon\omega_{2\delta}(x). \end{aligned}$$

Hence by (1.5.11) it follows that the inequality $u_\alpha(x) \leqq \Phi(x_0) - \varepsilon/3$ will hold in the intersection of D_0 with some sufficiently small neighborhood of x_0. On the other hand, by (1.5.9) and (1.5.10) the functions $u_\alpha(x)$ all satisfy the inequality

$$u_\alpha(x) \leqq \Phi(x_0) - 2\varepsilon/3 \tag{1.5.12}$$

throughout the set K'. Since the boundaries of the components $D_1, D_2, \cdots$ consist of points of K', it follows from the maximum principle that inequality (1.5.12) will also hold at all points of the domains $D_1, D_2, \cdots$. Consequently there is a neighborhood of x_0 such that at any point x of this neighborhood all the functions $u_\alpha(x)$ satisfy the inequality $u_\alpha(x) \leqq \Phi(x_0) - \varepsilon/3$. Hence it follows that $u(x) \leqq \Phi(x_0) - \varepsilon/3$ in this neighborhood of x_0, and consequently

$$u^*(x_0) \leqq \Phi(x_0) - \varepsilon/3 < \Phi(x_0) - \varepsilon/5.$$

This completes the proof of the lemma.

Now let the function $\Phi(x)$ figuring in the assumptions of Lemma 1.5.7 be $u^*(x)$. It then follows that if

$$\underline{\mathrm{Cap}}\, T_{\mathfrak{F}} > 0 \tag{1.5.13}$$

then there exists a point x_0 at which $u^*(x_0) < u^*(x_0) - \varepsilon/5$, for some $\varepsilon > 0$. This contradiction shows that inequality (1.5.13) cannot hold, so that the inner capacity of the set $T_{\mathfrak{F}}$ is zero. We now note that, by Lemma 1.5.5,

$$T_{\mathfrak{F}} \subset T_{\mathfrak{F}_0}, \tag{1.5.14}$$

where $\mathfrak{F}_0$ is a countable subfamily of $\mathfrak{F}$. Since we have proved the equality $\underline{\mathrm{Cap}}\, T_{\mathfrak{F}} = 0$ for any family $\mathfrak{F}$, it will also hold for the family $\mathfrak{F}_0$: $\underline{\mathrm{Cap}}\, T_{\mathfrak{F}_0} = 0$. On the other hand, it follows from Lemma 1.5.2 that $T_{\mathfrak{F}_0} \in G_{\delta\sigma}$, and this set is therefore capacitable. Consequently

$$\overline{\text{Cap}}\, T_{\mathfrak{F}_0} = \underline{\text{Cap}}\, T_{\mathfrak{F}_0} = 0.$$

Together with (1.5.14), this immediately implies that $\overline{\text{Cap}}\, T_{\mathfrak{F}} = 0$. We have thus proved

THEOREM 1.5.1 (H. CARTAN [1]). [20] *Let* $\mathfrak{F} = \{u_\alpha(x)\}$ *be a family of functions, subharmonic and uniformly bounded above in a domain* $G \subset R^m$. *Then the set* $T_{\mathfrak{F}}$ *of points* $x \in G$ *at which the function* $u(x) = \sup_{\mathfrak{F}} u_\alpha(x)$ *is strictly smaller than its regularization* $u^*(x)$ *satisfies the condition* $\text{Cap}\, T_{\mathfrak{F}} = 0$.

An analogous theorem holds for the set $S_{\mathfrak{F}}$:

THEOREM 1.5.2. *Let* $\mathfrak{F} = \{u_\alpha(x)\}_{\alpha \in (0,\infty)}$ *be a family of functions, subharmonic and uniformly bounded in a domain* $G \subset R^m$. *Then the set* $S_{\mathfrak{F}}$ *of all points* $x \in G$ *at which the function* $v(x) = \overline{\lim}_{\alpha \to \infty} u_\alpha(x)$ *is strictly smaller than its regularization* $v^*(x)$ *satisfies the condition* $\text{Cap}\, S_{\mathfrak{F}} = 0$.

PROOF. Set $v_n(x) = \sup_{\alpha \geqq n} u_\alpha(x)$, $n = 1,2,\cdots$, and consider the set D_ε^n of all points $x \in G$ at which $v_n(x) < v_n^*(x) - \varepsilon$. It follows from Theorem 1.5.1 that for arbitrary n and $\varepsilon > 0$

$$\text{Cap}\, D_\varepsilon^n = 0. \tag{1.5.15}$$

Note also that

$$v(x) = \lim_{n \to \infty} v_n(x), \tag{1.5.16}$$

and consider the set D' of all points $x \in G$ at which $v(x) < \lim_{n \to \infty} v_n^*(x)$. A direct check shows that

$$D' = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} D_{i/j}^n. \tag{1.5.17}$$

It follows from (1.5.17) and (1.5.15) that $\text{Cap}\, D' = 0$. We now claim that

$$v^*(x) = \lim_{n \to \infty} v_n^*(x).$$

Indeed, it follows from (1.5.16) that

$$v(x) \leqq \lim_{n \to \infty} v_n^*(x). \tag{1.5.18}$$

The functions $v_n^*(x)$ are subharmonic and constitute a monotone decreasing sequence. Hence the right-hand side of (1.5.18) is a subharmonic function; going from both sides of this inequality to their regularizations, we see that

[20] Our proof is different from that of Cartan [1].

$$v^*(x_0) \leqq \lim_{n\to\infty} v_n^*(x_0).$$

Suppose that at some point x_0 we have the strict inequality

$$v^*(x) < \lim_{n\to\infty} v_n^*(x).$$

Then, by the properties of subharmonic functions, the same inequality must hold for the averages of the functions $v^*(x)$ and $\lim_{n\to\infty} v_n^*(x)$ in a ball $E_r(x_0)$ of sufficiently small radius r. Consequently the inequality

$$v^*(x) < \lim_{n\to\infty} v_n^*(x)$$

is valid on some set of positive Lebesgue measure. On the same set, obviously, we have the inequality $v(x) < \lim_{n\to\infty} v_n^*(x)$. This contradicts the already proved equality Cap $D' = 0$. Consequently

$$v^*(x) = \lim_{n\to\infty} v_n^*(x)$$

and thus the set D' coincides with $S_{\mathfrak{F}}$. This immediately implies that Cap $S_{\mathfrak{F}} = 0$. Q.E.D.

6. Hartogs functions ($n = 1$)[21]

In this section we shall study the relation between subharmonic functions and analytic functions of one complex variable.

DEFINITION. A real-valued function $\Phi(z)$ defined in a domain $G \subset C^n$ is called a *Hartogs function* if it is in the intersection $\bigcap_{G'\Subset G} F_{G'}$, where $F_{G'}$ denotes the smallest class of functions, containing all functions $u(z) = \ln |f(z)|$ (where $f(z)$ is analytic in G'), which is closed under the following operations:

(1) formation of linear combinations $\Phi(z) = \lambda_1\Phi_1(z) + \lambda_2\Phi_2(z)$, with $\lambda_1, \lambda_2 > 0$;

(2) formation of least upper bounds $\Phi(z) = \sup_\alpha \Phi_\alpha(z)$, where $\{\Phi_\alpha(z)\}$ is a family of functions uniformly bounded above on every compact set $K \subset G'$;

(3) formation of limits $\Phi(z) = \lim_{j\to\infty} \Phi_j(z)$, where the sequence $\{\Phi_j(z)\}$ is monotone decreasing.

It follows from the results of § 1 that any upper semicontinuous Hartogs function is subharmonic. For $n = 1$, the converse is also true:

THEOREM 1.6.1 (LELONG [1]). *A function $u(z)$ subharmonic in a domain $G \subset C^1$ is a Hartogs function.*

PROOF. Let $G' \Subset G$ and let μ be the associated measure of the function $u(z)$. Partition the plane C^1 by straight lines parallel to the coordinate axes into open squares with sides of length 2^{-m}. Since the measure μ is finite in any domain G', the number of straight lines carrying positive mass is at most countable. Consequently we can always displace the above lattice so that the straight lines all have zero measure. Denote the squares which are not disjoint from G' by β_i^m, $i = 1,\cdots,N_m$.

[21] The results of this section are not used in the sequel, and the reader desiring a rapid introduction to entire functions may safely proceed to § 7. Nevertheless, an acquaintance with this section and the related § 5 of Chapter 2 is useful in that it clarifies the reasons why subharmonic functions of two (real) variables, unlike subharmonic functions of more variables, play such an important role in the theory of analytic functions.

Since $G' \Subset G$, it follows that $N_m < \infty$; in fact, $N_m < 4^{m+1}d^2$, where d is the diameter of the domain G'. In each square β_i^m, choose a point $\zeta_i \in G'$ and place there a mass

$$\frac{1}{5^m}[5^m \mu(\beta_i^m \cap G')].^{22)}$$

The mass distribution thus defined in G' will be denoted by μ_m.

LEMMA 1.6.1. *For any function $\varphi(z)$ continuous in G,*

$$\lim_{m\to\infty} \int_{G'} \varphi(z)\,d\mu_m(z) = \int_{G'} \varphi(z)\,d\mu.$$

PROOF. Since the function $\varphi(z)$ is continuous on the compact set $\bar{G}'$, for any $\varepsilon > 0$ there exists a number $m_0 = m_0(\varepsilon)$ such that the variation of the function $\varphi(z)$ on each compact set $\overline{\beta_i^m \cap G'}$ is at most ε for $m \geqq m_0$. Consequently, for $m \geqq m_0$ we have

$$\begin{aligned}
\left| \int_{G'} \varphi d\mu_m - \int_{G'} \varphi d\mu \right| &= \left| \sum_{i=1}^{N_m} \left\{ \int_{\beta_i^m \cap G'} \varphi\, d\mu_m - \int_{\beta_i^m \cap G'} \varphi\, d\mu \right\} \right| \\
&= \left| \sum_{i=1}^{N_m} \left\{ \varphi(\zeta_i)\mu_m(\beta_i^m \cap G') + \int_{\beta_i^m \cap G'} (\varphi(\zeta_i) - \varphi(z))\,d\mu - \varphi(\zeta_i)\mu(\beta_i^m \cap G') \right\} \right| \\
&\leqq \varepsilon\mu(G') + \max_i |\varphi(\zeta_i)| \cdot \sum_{i=1}^{N_m} |\mu_m(\beta_i^m \cap G') - \mu(\beta_i^m \cap G')| \\
&\leqq \varepsilon\mu(G') + \max_i |\varphi(\zeta_i)| \frac{4^{m+1}d^2}{5^m}.
\end{aligned}$$

Since ε is arbitrary, it follows immediately that

$$\lim_{m\to\infty} \left| \int_{G'} \varphi\, d\mu_m - \int_{G'} \varphi\, d\mu \right| = 0.$$

Q.E.D.

Now consider the integral

$$v_\mu^t(z) = \int_{G'} h_t(z - \zeta)\,d\mu_\zeta,$$

where $h_t(z) = \max\{\ln|z|, t\}$, and the potential

$$v_\mu(z) = \int_{G'} \ln|z - \zeta|\,d\mu_\zeta.$$

The functions $v_{\mu_m}^t$ and v_{μ_m} are defined similarly. Note that by Lemma 1.6.1

$$\lim_{m\to\infty} v_{\mu_m}^t(z) = v_\mu^t(z).$$

Hence, in view of the obvious inequality $v_{\mu_m}^t(z) \geqq v_{\mu_m}(z)$, we conclude that

$$\overline{\lim_{m\to\infty}}\, v_{\mu_m}(z) \leqq v_\mu^t(z).$$

Letting $t \to -\infty$ in this inequality, we see that for $z \in G'$

$$\overline{\lim_{m\to\infty}}\, v_{\mu_m}(z) \leqq v_\mu(z).$$

On the other hand, for any $\varepsilon > 0$ the inequality

$$\overline{\lim_{m\to\infty}}\, v_{\mu_m}(z) > v_\mu(z) - \varepsilon \tag{1.6.1}$$

---

22) As usual, $[x]$ denotes the integral part of x.

holds on a set which is dense in G', Indeed, if the inequality

$$\overline{\lim_{m\to\infty}}\, v_{\mu_m}(z) \leqq v_\mu(z) - \varepsilon \tag{1.6.2}$$

holds at all points of some disk $E_r(z_0) \subset G'$, then, by Fatou's Lemma, for any function $\varphi(z)$, continuous in $E_{r/2}(z_0)$ and with compact support in $E_r(z_0)$,

$$\overline{\lim_{m\to\infty}} \int_{G'} \int_{G'} \varphi(z) \ln|z-\zeta|\, d\mu_m(\zeta)\, d\omega_z = \overline{\lim_{m\to\infty}} \int_{G'} \varphi(z) v_{\mu_m}(z) d\omega_z$$

$$\leqq \int_{G'} \varphi(z) \left\{ \overline{\lim_{m\to\infty}}\, v_{\mu_m}(z) \right\} d\omega_z \leqq \int_{G'} \varphi(z) v_\mu(z) d\omega_z - \varepsilon \int_{G'} \varphi(z) d\omega_z < \int_{G'} \varphi(z) v_\mu(z)\, d\omega_z.$$

But since the function $\int_{G'} \ln|z-\zeta|\, \varphi(z)\, d\omega_z$ is clearly continuous in ζ, it follows from Lemma 1.6.1 that

$$\lim_{m\to\infty} \int_{G'} \int_{G'} \varphi(z) \ln|z-\zeta|\, d\mu_m(\zeta)\, d\omega_z = \lim_{m\to\infty} \int_{G'} d\mu_m(\zeta) \int_{G'} \varphi(z) \ln|z-\zeta|\, d\omega_z = \int_{G'} \varphi(z) v_\mu(z)\, d\omega_z.$$

This contradiction shows that there is no disk $E_r(z_0)$ in which inequality (1.6.2) holds. Consequently for any $\varepsilon > 0$ inequality (1.6.1) holds on a set dense in G', whence it follows immediately that the regularization $v^*(z)$ of the function

$$v(z) = \overline{\lim_{m\to\infty}}\, v_{\mu_m}(z) \tag{1.6.3}$$

coincides with the function $v_\mu(z)$ in G'. Note that every function $v_{\mu_m}(z)$ is in class $F_{G'}$. Indeed, by the definition of the measure μ_m,

$$v_{\mu_m}(z) = \int_{G'} \ln|z-\zeta|\, d\mu_m(\zeta) = \frac{1}{5^m} \sum_{i=1}^{N_m} k_i \ln|z-\zeta_i|,$$

where $k_i = [5^m \mu(\beta_i^m \cap G')]$.

Now, as we have just seen, $v^* = v_\mu$, and moreover the regularization is constructed by means of operations under which the class $F_{G'}$ is closed (cf. the definition of Hartogs functions); thus the fact that the functions $v_{\mu_m}(z)$ are in $F_{G'}$ implies that the potential $v_\mu(z)$ is also in $F_{G'}$. We claim that any function $\Phi(z)$ harmonic in G' is in $F_{G'}$. Since every domain G is the union of finitely connected domains, it will suffice to prove this assertion for the case of a finitely connected domain G'. In this case there obviously exist real numbers $\lambda_1, \cdots, \lambda_p$ and complex numbers $a_2 \notin G', \cdots, a_p \notin G'$ such that the function

$$\Phi_1(z) = \Phi(z) - \sum_{j=1}^{p} \lambda_j \ln|z-a_j|,$$

which is harmonic in G', is the real part in G' of some univalent analytic function. Therefore $\Phi_1 \in F_{G'}$, which immediately implies that $\Phi \in F_{G'}$. Thus both terms in the Riesz decomposition of the subharmonic function $u(z)$ (Theorem 1.2.1') are in $F_{G'}$, and hence so is $u(z)$ itself. Since the domain G is the union of domains G', it follows that $u(z)$ is indeed a Hartogs function. Q.E.D.

Theorem 1.6.1 fails to hold for severable variables. In the next chapter we shall prove that in the general case the upper semicontinuous Hartogs functions form a special subclass in the class of subharmonic functions.

§ 7. Phragmén-Lindelöf principle

The simple examples cited in § 1 of this chapter show that the maximum principle fails for unbounded domains. Nevertheless, it can be shown that the maximum principle remains valid for unbounded domains, provided certain restrictions are imposed on the growth of the subharmonic functions in question. This is one

result of the following theorem from the theory of analytic functions of one variable, which is known as the Phragmén-Lindelöf principle for an angle.[23]

THEOREM 1.7.1. *Let* $G_\alpha = \{z: z \in C^1,\ 0 < \arg z < \alpha\pi\}$ $(0 < \alpha < 2)$, *and let* $f(z)$ *be a function analytic in* G_α, *satisfying the following conditions*:

(a) $$\overline{\lim_{\substack{z\to\zeta \\ z\in G_\alpha}}} |f(z)| \leqq C \qquad \forall \zeta \in \partial G_\alpha,$$

(b) $$\overline{\lim_{r\to\infty}}\, r^{-1/\alpha} \ln M_f(r) \leqq 0,$$

where

$$M_f(r) = \sup_{\substack{|z|=r \\ 0<\arg z<\alpha\pi}} |f(z)|.$$

Then $|f(z)| \leqq C$ *everywhere in the angle* G_α, *and the equality* $|f(z_0)| = C$ *can hold for* $z_0 \in G_\alpha$ *only if* $f(z) \equiv C$.

To generalize this theorem to functions subharmonic in a cone in $R^m_{(x)}$, we introduce "spherical" coordinates in the space:

$$\begin{aligned}
x_1 &= \rho \cos\theta_1 \\
x_2 &= \rho \cos\theta_2 \sin\theta_1 \\
&\cdots\cdots\cdots\cdots \\
x_i &= \rho \cos\theta_i \sin\theta_1 \cdots \sin\theta_{i-1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_m &= \rho \sin\theta_1 \cdots \sin\theta_{m-1},
\end{aligned}$$

where $0 \leqq \theta_1 \leqq \pi$, $0 \leqq \theta_2 \leqq 2\pi, \cdots, 0 \leqq \theta_{m-1} < 2\pi$.

The Laplacian has the following form in these coordinates:[24]

$$\Delta_{\theta,\rho} = \frac{\partial^2}{\partial\rho^2} + \frac{m-1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\Delta^*_\theta, \tag{1.7.1}$$

where

$$\Delta^*_\theta = \left\{\frac{\partial^2}{\partial\theta_1^2} + (m-2)\cot\theta_1 \frac{\partial}{\partial\theta_1}\right\} + \frac{1}{\sin^2\theta_1}\left\{\frac{\partial^2}{\partial\theta_2^2} + (m-3)\cot\theta_2 \frac{\partial}{\partial\theta_2}\right\} + \cdots$$
$$+ \frac{1}{\prod_{j=1}^{i-1}\sin^2\theta_j}\left\{\frac{\partial^2}{\partial\theta_j^2} + (n-1-i)\cot\theta_i \frac{\partial}{\partial\theta_i}\right\} + \frac{1}{\prod_{j=1}^{m-2}\sin^2\theta_j}\frac{\partial^2}{\partial\theta_{m-1}^2}.$$

It follows from (1.7.1) that the product $\varphi(\theta)\alpha(\rho)$ is a harmonic function if and only if the function $\varphi(\theta)$ is a solution of the equation

[23] See, for example, Markuševič [1].
[24] See, for example, Madelung [1].

$$\Delta_\theta^* \varphi + \lambda \varphi = 0, \tag{1.7.2}$$

and the function $\alpha(\rho)$ a solution of the equation

$$\rho^2 \alpha'' + (m-1)\rho\alpha' - \lambda\alpha = 0 \tag{1.7.3}$$

with the same constant λ as in (1.7.2). Note that the functions

$$\alpha(\rho) = c_1 \rho^{\mu_1} + c_2 \rho^{\mu_2}, \tag{1.7.4}$$

where

$$\mu_{1,2} = \frac{-(m-2) \pm \sqrt{(m-2)^2 + 4\lambda}}{2},$$

are solutions of equation (1.7.3).

The following theorem is a natural extension of the Phragmén-Lindelöf principle to functions subharmonic in a cone.

THEOREM 1.7.2 (DENY-LELONG [1]).[25] *Let G be a domain on the unit sphere $S_1(0)$ in the space R^m, whose boundary ∂G belongs to some class $A^{(2,\lambda)}$[26] in the space of variables $\theta_1, \cdots, \theta_{m-1}$. Assume, moreover, that, after a suitable displacement, G is relatively compact in the domain $\{\theta: 0 < \theta_i < \pi,\ i = 1, \cdots, m-1\}$. Let the function $u(x) = u(\theta, \rho)$ be subharmonic in the cone $V = \{(\theta, \rho): \theta \in G,\ 0 < \rho < \infty\}$ and assume moreover that*

(a) $$\overline{\lim_{x \to x_0,\ x \in V}} u(x) \leqq 0 \qquad \forall x_0 \in \partial V,$$

(b) $$\overline{\lim_{\rho \to \infty}} \rho^{-\mu_1} M_u(\rho) \leqq 0,$$

where

$$M_u(\rho) = \sup_{\theta \in G} u(\theta, \rho), \qquad \mu_1 = \frac{-(m-2) + \sqrt{(m-2)^2 + 4\lambda_1}}{2},$$

and λ_1 is the minimal eigenvalue of the equation (1.7.2) *in the domain G with boundary condition*

$$\varphi(\theta) = 0 \qquad \forall\, \theta \in \partial G_1. \tag{1.7.5}$$

Then $u(x) \leqq 0$ for all $x \in V$, and equality can hold at some point $x_0 \in V$ only if $u(x) \equiv 0$.

PROOF. Consider a function $\Phi(\theta, \rho)$, harmonic in the domain $V_1 = V \cap E_1(0)$, continuous in $\bar{V}_1 \setminus \{S_1(0) \cap \partial V\}$, and such that

[25] The statement of this theorem in Deny-Lelong [1] is slightly different.

[26] A surface $S \subset R^l$, $l \geqq 2$, belongs to class $A^{(2,\lambda)}$ if it is defined in the neighborhood of each point, in a suitable coordinate system $\xi_1, \cdots, \xi_l$, by an equation $\xi_l = \psi(\xi_1, \cdots, \xi_{l-1})$, where the function $\psi(\xi_1, \cdots, \xi_{l-1})$ has second derivatives that satisfy a Hölder condition with exponent λ. For $l = 1$, any finite set of points belongs by definition to $A^{(2,\lambda)}$. See, for example, Miranda [1].

$$\Phi(\theta, \rho) = 0 \qquad \forall \theta \in \partial G, \qquad 0 < \rho < 1,$$
$$\Phi(\theta, 1) = 1 \qquad \forall \theta \in \partial G.$$

It follows from estimates for the derivatives of the solutions of the Dirichlet problem[27] and from general theorems on eigenfunction expansions[28] that the functions $\Phi(\theta, \delta)$, $0 < \delta < 1$, admit the representation

$$\Phi(\theta, \delta) = \sum_{n=1}^{\infty} a_n \varphi_n(\theta) \qquad (a_n = a_n(\delta)), \tag{1.7.6}$$

where $\varphi_n(\theta)$ are the eigenfunctions of equation (1.7.2) in the domain G with boundary condition (1.7.5). The series (1.7.6) is absolutely and uniformly convergent in $\bar{G}$. The eigenvalues belonging to the functions $\varphi_n(\theta)$ are denoted by λ_n ($\lambda_1 < \lambda_2 < \cdots$). Set

$$\mu_n = \frac{-(m-2) + \sqrt{(m-2)^2 + 4\lambda_n}}{2}$$

and consider the function

$$\Phi_\delta(\theta, \rho) = \sum_{n=1}^{\infty} a_n \left(\frac{\rho}{\delta}\right)^{\mu_n} \varphi_n(\theta). \tag{1.7.7}$$

The series in (1.7.7) is obviously absolutely and uniformly convergent for $0 < \rho < \delta$. As follows from the remarks preceding Theorem 1.7.2, each term of this series is a harmonic function. Consequently the function $\Phi_\delta(\theta, \rho)$ is harmonic in the domain $V_\delta = V \cap E_\delta(0)$. Taking into account the obvious equalities

$$\Phi_\delta(\theta, \rho) = 0 \qquad \forall \theta \in \partial G, \qquad 0 < \rho < \delta,$$
$$\Phi_\delta(\theta, \delta) = \Phi(\theta, \delta) \qquad \forall \theta \in G$$

and the fact that the functions $\Phi(\theta, \rho)$ and $\Phi_\delta(\theta, \rho)$ are bounded in V_δ, we conclude that $\Phi_\delta(\theta, \rho) = \Phi(\theta, \rho)$ in V_δ. Together with (1.7.7), this implies that for $\rho \leqq \delta$

$$\Phi(\theta, \rho) = \sum_{n=1}^{\infty} a_n \left(\frac{\rho}{\delta}\right)^{\mu_n} \varphi_n(\theta). \tag{1.7.8}$$

We now estimate the rate of convergence to zero of the function $\Phi(\theta, \rho)$ as $\rho \to 0$. We have

$$\Phi(\theta, \rho) = \left(\frac{\rho}{\delta}\right)^{\mu_1} \left\{ a_1 \varphi_1(\theta) + \sum_{n=2}^{\infty} a_n \varphi_n(\theta) \left(\frac{\rho}{\delta}\right)^{\mu_n - \mu_1} \right\}.$$

Since $\lambda_n > \lambda_1$ for any n,[29] it follows that $\mu_n > \mu_1$, and hence

[27] See, for example, Miranda [1].

[28] See, for example, Courant and Hilbert [1].

[29] The fact that the first eigenvalue λ_1 is simple follows from general theorems on eigenvalues and eigenfunctions of elliptic operators; see, for example, Courant and Hilbert [1].

$$\sup_{(\theta,\rho)\in V_\delta}\left|\left\{a_1\varphi_1(\theta)+\sum_{n=2}^{\infty}a_n\left(\frac{\rho}{\delta}\right)^{\mu_n-\mu_1}\varphi_n(\theta)\right\}\right|<\infty.$$

Consequently the inequality

$$\Phi(\theta,\rho)\leqq A\rho^{\mu_1} \tag{1.7.9}$$

will hold in V_1 for some constant $A<\infty$.

Now let $0<\rho<r$, and consider the function

$$H_r(\theta,\rho)=u(\theta,\rho)-M_u(r)\,\Phi(\theta,\rho/r).$$

This function is clearly subharmonic in V_r, and, moreover,

$$\overline{\lim_{\substack{(\theta,\rho)\to(\theta',\rho')\\(\theta,\rho)\in V_r}}}H_r(\theta,\rho)\leqq 0\qquad\forall(\theta',\rho')\in\partial V_r\backslash\{S_r(0)\cap\partial V\}. \tag{1.7.10}$$

By the generalized maximum principle, it follows from (1.7.10) that $H_r(\theta,\rho)\leqq 0$ throughout V_r. Consequently, for any $r>\rho$ and any $\theta\in G$,

$$u(\theta,\rho)\leqq M_u(r)\Phi(\theta,\rho/r).$$

Together with (1.7.9), this implies that

$$u(\theta,\rho)\leqq AM_\mu(r)\rho^{\mu_1}/r^{\mu_1}.$$

Further, using condition (b) of the theorem, we see that

$$\overline{\lim_{r\to\infty}}\,u(\theta,\rho)\leqq A\rho^{\mu_1}\,\overline{\lim_{r\to\infty}}\,\frac{M_u(r)}{r^{\mu_1}}\leqq 0,$$

and therefore at any point $(\theta,\rho)\in V$ we have the inequality

$$u(\theta,\rho)\leqq 0. \tag{1.7.11}$$

It now follows from the maximum principle that equality can hold in (1.7.11) only if $u(\theta,\rho)\equiv 0$. Q.E.D.

The Phragmén-Lindelöf principle for a cylinder is proved in analogous fashion:

THEOREM 1.7.3. *Let G be a bounded domain in R^{m-1}, with boundary ∂G belonging to some class $A^{(2,\lambda)}$. Let $V=\{x=(x',x_m)\colon x'\in G,\ x_m\in R^1_+\}$, and let $u(x)$ be a function subharmonic in the cylinder V such that*

(a)
$$\overline{\lim_{\substack{x\to x_0\\x\in V}}}\,u(x)\leqq 0\qquad\forall x_0\in\partial V,$$

(b)
$$\overline{\lim_{h\to+\infty}}\,\frac{m_u(h)}{e^{h\sqrt{\lambda_1}}}\leqq 0,$$

where

$$m_u(h)=\sup_{x'\in G}u(x',h),$$

λ_1 being the minimal eigenvalue of the equation $\Delta_{x'} f + \lambda f = 0$ in the domain G with boundary condition $f(x') = 0$ $\forall x' \in \partial G$.

Then $u(x) \leqq 0$ throughout the interior of V, and equality can hold at some point $x_0 \in V$ only if $u(x) \equiv 0$.

It follows from Theorem 1.7.2 that the Phragmén-Lindelöf principle for an angle can be formulated as follows:

THEOREM 1.7.4. *Let the function $u(z)$, $z \in C^1$, be subharmonic in the angle $G_\alpha = \{z: z \in C^1,\ 0 < \arg z < \alpha\pi\},\ 0 < \alpha < 2$, and such that*

(a) $$\overline{\lim_{\substack{z \to \zeta \\ z \in G_\alpha}}} u(z) \leqq c \qquad \forall \zeta \in \partial G_\alpha,$$

(b) $$\overline{\lim_{r \to \infty}} \frac{M_u(r)}{r^{1/\alpha}} \leqq 0.$$

Then $u(z) \leqq c$ for all $z \in G_\alpha$, and equality $u(z_0) = c$ can hold for $z_0 \in G_\alpha$ only if $u(z) \equiv c$.

To prove this theorem, it is sufficient to apply Theorem 1.7.2 to the function $u(z) - c$, recalling that in the case $m = 2$ we have $\Delta_\theta^* = \partial^2/\partial\theta_1^2$ and, as is readily seen, $\lambda_1 = 1/\alpha^2$ and $\mu_1 = 1/\alpha$ for the domain G_α.

To conclude this section, we prove a theorem which is one of the versions of the Phragmén-Lindelöf principle for a half-plane.

THEOREM 1.7.5. *Let $u(z)$, $z \in C^1$, be a function subharmonic in the upper half-plane $E = \{z: \operatorname{Im} z > 0\}$, with the following property: there exists a real number σ such that for any $\varepsilon > 0$ there exist constants A_ε and C_ε for which the inequality*

$$u(z) \leqq C_\varepsilon + (\sigma + \varepsilon)\, y + A_\varepsilon |x|$$

holds for any $z = x + iy$, $y > 0$. Assume moreover that

$$\overline{\lim_{\substack{z \to x \\ z \in E}}} u(z) \leqq M \qquad \forall x \in \partial E.$$

Then $u(z) \leqq M + \sigma y$ throughout E.

PROOF. Consider the function $u_\varepsilon(z) = u(z) - (\sigma + \varepsilon)y$. It is clear that $u_\varepsilon(z)$ is subharmonic in E. In each of the quadrants $E \cap \{z: \operatorname{Re} z < 0\}$ and $E \cap \{z: \operatorname{Re} z > 0\}$ this function satisfies the assumptions of Theorem 1.7.4 with $u_\varepsilon(iy) \leqq C_\varepsilon$ $\forall y > 0$, and

$$\overline{\lim_{z \to x,\ z \in E}} u_\varepsilon(z) \leqq M \qquad \forall x \in \partial E.$$

Consequently

$$\sup_E u_\varepsilon(z) \leqq \max\{M, C_\varepsilon\} < \infty. \tag{1.7.12}$$

We claim that a stronger inequality is valid:

$$\sup_{z\in E} u_\varepsilon(z) \leqq M.$$

To prove this, consider the function $\Phi_\eta(z) = u_\varepsilon(z) - \ln|\eta z + i|$. For $\eta > 0$, this function is obviously subharmonic in E. Moreover, it follows from (1.7.12) that

$$\lim_{|z|\to\infty} \Phi_\eta(z) = -\infty.$$

Therefore, as is easily seen,

$$\sup_E \Phi_\eta(z) = \sup_{-\infty<x<\infty} \overline{\lim_{\substack{z\to x\\ z\in E}}} \Phi_\eta(z) \leqq \sup_{-\infty<x<\infty} \overline{\lim_{\substack{z\to x\\ z\in E}}} u_\varepsilon(z).$$

Thus, for any $\varepsilon > 0$, $\eta > 0$ and $z \in E$,

$$u_\varepsilon(z) - \ln|\eta z + i| \leqq M.$$

Letting $\eta \to 0$ in this inequality, we get $u_\varepsilon(z) \leqq M$ for all $z \in E$. Consequently $u(z) \leqq M + (\sigma + \varepsilon)y$ throughout E. Hence, since ε is arbitrary, it follows that $u(z) \leqq M + \sigma y$ for any $z \in E$. Q.E.D.

CHAPTER 2

PLURISUBHARMONIC FUNCTIONS

In this chapter we study a special class of subharmonic functions, which are related more closely than subharmonic functions to analytic functions of several variables.

§ 1. General theory of plurisubharmonic functions

1. *Definition. Examples.* Several equivalent definitions of plurisubharmonic functions are known. The following definition is the most widely used.

DEFINITION. A function $u(z)$, $-\infty \leqq u(z) < \infty$, defined in a domain $G \subset C^n$, is said to be *plurisubharmonic* if it is upper semicontinuous and for any complex numbers $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ the function $u(aw + b) = u(a_1w + b_1, \cdots, a_nw + b_n)$ is a subharmonic function of the complex scalar w in every component of the open set $\{w: (a_1w + b_1, \cdots, a_nw + b_n) \in G\}$.[1)]

It follows from this definition that if $f(z)$ is an analytic function in a domain $G \subset C^n$, then the functions $\ln |f(z)|$, $\operatorname{Re} f(z)$ and $|f(z)|^\alpha$, $\alpha > 0$, are plurisubharmonic in G.

2. *Simplest properties of plurisubharmonic functions.* The following propositions follow immediately from the definition and from the properties of subharmonic functions considered in Chapter 1.

(a) *The product of a plurisubharmonic function and a positive constant is again a plurisubharmonic function.*

(b) *The sum of finitely many plurisubharmonic functions is again a plurisubharmonic function.*

(c) *If the function $u(z)$ is plurisubharmonic in a domain $G \subset C^n$, then on every component of the set $\{z: z_{j+1} = z_{j+1}^0, \cdots, z_n = z_n^0\} \cap G$ the function*

$$u(z_1, \cdots, z_j, z_{j+1}^0, \cdots, z_n^0)$$

is a plurisubharmonic function of the variables $z_1, \cdots, z_j$.

[1)] An equivalent formulation of this definition is as follows: *The restriction of the function $u(z)$ to any complex one-dimensional analytic plane is a subharmonic function in each component of the intersection of the domain with the plane.*

(d) *If the functions* $u_1(z), \cdots, u_N(z)$ *are plurisubharmonic in a domain* $G \subset C^n$, *then the function*

$$u(z) = \max_{1 \leq j \leq N} u_j(z)$$

is also plurisubharmonic in G.

(e) *Let* $\{u_\alpha(z)\}$ *be a family of functions, plurisubharmonic in a domain* $G \subset C^n$, *and let the function* $u(z) = \sup_\alpha u_\alpha(z)$ *be upper semicontinuous. Then* $u(z)$ *is also plurisubharmonic in* G.

(f) *The limit of a uniformly convergent sequence of plurisubharmonic functions is a plurisubharmonic function.*

(g) *The limit of a monotone decreasing sequence of plurisubharmonic functions is a plurisubharmonic function.*

As in Chapter 1, properties (f) and (g) are also valid for convergence with respect to a continuous parameter.

(h) MAXIMUM PRINCIPLE. *A function* $u(z)$ *plurisubharmonic in a domain* $G \Subset C^n$ *can take on the value* $\sup_G u(z)$ *in the interior of* G *only if* $u(z) \equiv$ const.

(i) INTEGRATION WITH RESPECT TO A PARAMETER. *Let* $G \subset C^n$, $D \subset R^m$ *and* $D_0 \Subset D$ *be domains, and* μ *a measure on* D. *Let the function* $u(z, \lambda)$ *be upper semicontinuous in the domain* $G \times D$ *and plurisubharmonic in* G *for each fixed* $\lambda \in D$. *Then the function*

$$V(z) = \int_{D_0} u(z, \lambda)\, d\mu_\lambda$$

is plurisubharmonic in G. Note that here, as in § 1.5 of Chapter 1, integration over D_0 may be replaced by integration over D, provided $\mu(D) < \infty$.

(j) *Let* $\varphi(t)$ *be a convex monotone nondecreasing function of the variable* $t \in R^1$, *and* $u(z)$ *a plurisubharmonic function in a domain* $G \subset C^n$ *such that* $\varphi(u(z))$ *is defined for all* $z \in G$. *Then the function* $\Phi(z) = \varphi(u(z))$ *is plurisubharmonic.*

In particular, if the logarithm $\ln u(z)$ of a nonnegative function $u(z)$ is a plurisubharmonic function, then $u(z)$ is also plurisubharmonic. Such functions are said to be *logarithmically plurisubharmonic*. It is clear that they possess properties analogous to those of logarithmically subharmonic functions. Thus logarithmic plurisubharmonicity is preserved under addition and under integration with respect to a parameter (cf. (d) and (e) in § 2.2 of Chapter 1).

3. *Averages.* Let $u(z)$ be a plurisubharmonic function in a domain $G \subset C^n$ and assume that the polydisk $\mathscr{P}(z^0, r) = \{z: |z_i - z_i^0| < r_i,\ i = 1, \cdots, n\}$ is contained in G. Let $\mathfrak{N}_u^c(z^0, r)$ denote the average of the function $u(z)$ over the polydisk $\mathscr{P}(z^0, r)$:

$$\mathfrak{N}_u^c(z^0, r) = \frac{1}{\pi^n r_1^2 \cdots r_n^2} \int_{\mathscr{P}(z^0, r)} u(z)\, d\omega_z.$$

The average of the function $u(z)$ over the distinguished boundary of the polydisk $\mathscr{P}(z^0, r)$ is denoted by $\mathfrak{M}_u^c(z^0, r)$:

$$\mathfrak{M}_u^c(z^0, r) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} u(z_1^0 + r_1 e^{i\theta_1}, \cdots, z_n^0 + r_n e^{i\theta_n})\, d\theta_1 \cdots d\theta_n.$$

The properties of the functions $\mathfrak{N}_u^c$ and $\mathfrak{M}_u^c$ are in the main analogous to those of the functions $\mathfrak{N}_u$ and $\mathfrak{M}_u$ (see § 1.4 of Chapter 1), as the following propositions show.

(a) *The function $\mathfrak{N}_u^c$ and $\mathfrak{M}_u^c$ satisfy the following relation*:

$$\mathfrak{N}_u^c(z, r) = \frac{2^n}{r_1^2 \cdots r_n^2} \int_0^{r_1} \cdots \int_0^{r_n} t_1 t_2 \cdots t_n \mathfrak{M}_u^c(z, t)\, dt_1 \cdots dt_n. \tag{2.1.1}$$

This is verified by a direct check.

(b) *The functions $\mathfrak{N}_u^c(z, r)$ and $\mathfrak{M}_u^c(z, r)$ are monotone nondecreasing in each of their variables.*

Here we prove monotonicity in the variable r_1. We have

$$\begin{aligned}\mathfrak{N}_u^c(z^0, r) &= \frac{(i/2)^n}{\pi^n r_1^2 \cdots r_n^2} \int_{\mathcal{B}(z^0, r)} u(z)\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= \frac{(i/2)^{n-1}}{\pi^{n-1} r_2^2 \cdots r_n^2} \int_{\substack{|z_i - z_i^0| < r_i \\ i=2, \cdots, n}} \left\{ \frac{i/2}{\pi r_1^2} \int_{|z_1 - z_1^0| < r_1} u(z) dz_1 \wedge d\bar{z}_1 \right\} dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n.^{2)}\end{aligned}$$

For any fixed z_i, $i = 2, \cdots, n$, such that $|z_i - z_i^0| < r_i$, the inner integral on the right is the average over the disk $|z_1 - z_1^0| < r_1$ of the function $\varphi(z_1) = u(z_1, \cdots, z_n)$, which is subharmonic in this disk. The integral in question is therefore a monotone nondecreasing function of r_1 for such values of $z_2, \cdots, z_n$. Consequently the same holds for the function $\mathfrak{N}_u^c(z^0, r)$.

The monotonicity of $\mathfrak{M}_u^c(z^0, r)$ is proved in analogous fashion.

(c) *The functions $u(z)$, $\mathfrak{M}_u^c(z, r)$ and $\mathfrak{N}_u^c(z, r)$ satisfy the inequalities*

$$u(z) \leqq \mathfrak{N}_u^c(z, r) \leqq \mathfrak{M}_u^c(z, r). \tag{2.1.2}$$

The proof of these inequalities is based on the analogous property (1.1.8) of subharmonic functions. From this property we conclude, since the function $u(z)$ is subharmonic in z_1, that

$$\begin{aligned}u(z_1^0, z_2, \cdots, z_n) &\leqq \frac{i}{2\pi r_1^2} \int_{|z_1 - z_1^0| < r_1} u(z)\, dz_1 \wedge d\bar{z}_1 \\ &\leqq \frac{i}{2\pi} \int_0^{2\pi} u(z_1^0 + r_1 e^{i\theta_1}, z_2, \cdots, z_n)\, d\theta_1.\end{aligned} \tag{2.1.3}$$

[2)] All exterior differential forms appearing in the text are assumed to be odd, and hence their integral is invariant under transformations of the coordinates. Thus the form

$$(i/2)^n\, dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$$

is the volume element in the space C^n.

Both integrals in (2.1.3) are obviously plurisubharmonic functions of the variables $z_2, \cdots, z_n$. We can therefore apply inequality (1.1.8) for the variable z_2 to all terms in (2.1.3). We get

$$u(z_1^0, z_2^0, z_3, \cdots, z_n) \leqq \frac{(i)^2}{4\pi^2 r_1^2 r_2^2} \int\limits_{\substack{|z_i - z_i^0| < r_i \\ i=1,2}} u(z)dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$

$$\leqq \frac{1}{4\pi^2} \int_0^{2\pi} \cdots \int_0^{2\pi} u(z_1^0 + r_1 e^{i\theta_1}, z_2^0 + r_2 e^{i\theta_2}, z_3, \cdots, z_n)\, d\theta_1 d\theta_2.$$

Repeating this argument as many times as required, we arrive at inequality (2.1.2).

(d) *For any function $u(z)$ plurisubharmonic in a domain $G \subset C^n$,*

$$\lim_{|r| \to 0} \mathfrak{M}_u^c(z, r) = u(z), \qquad \lim_{|r| \to 0} \mathfrak{N}_u^c(z, r) = u(z).$$

The proofs of these inequalities are omitted, since they are almost the same, word for word, as those of the corresponding inequalities (c) of § 1.4, Chapter 1, for subharmonic functions.

(e) *If the function $u(z) \not\equiv -\infty$ is plurisubharmonic in a domain G, then $\mathfrak{N}_u^c(z, r) > -\infty$ for all z and r for which the function $\mathfrak{N}_u^c(z, r)$ is defined. In other words, a plurisubharmonic function which is not identically $-\infty$ is locally integrable.*

The proof is omitted, for the same reasons as before.

(f) *For fixed r the functions $\mathfrak{N}_u^c(z, r)$ and $\mathfrak{M}_u^c(z, r)$ are plurisubharmonic functions of $z_1, \cdots, z_n$ in the open set $\{z: \mathscr{E}(z, r) \Subset G\}$.*

This proposition is a direct corollary of the fact, proved above, that plurisubharmonicity is preserved under integration with respect to a parameter.

4. *Approximation of plurisubharmonic functions.* As is the case for subharmonic functions, any plurisubharmonic function can be approximated by infinitely differentiable plurisubharmonic functions. The exact result is as follows:

Let $u(z)$ be a function plurisubharmonic in some domain $G \subset C^n$. Let $\{G_\varepsilon\}$ be a family of domains such that the polydisk $\mathscr{E}(z, \varepsilon, \cdots, \varepsilon)$ is in G for any $z \in G_\varepsilon$, $G_{\varepsilon_1} \subset G_{\varepsilon_2}$ for $\varepsilon_1 > \varepsilon_2$, and $\bigcup_\varepsilon G_\varepsilon = G$. Then there exists a family of functions $u_\varepsilon(z)$ satisfying the following conditions.

(a) *For every $\varepsilon > 0$ the function $u_\varepsilon(z)$ is infinitely differentiable and plurisubharmonic in G.*

(b) *For $\varepsilon_1 > \varepsilon_2$ and $z \in G_{\varepsilon_1}$, $u_{\varepsilon_1}(z) \geqq u_{\varepsilon_2}(z)$.*

(c) $\lim_{\varepsilon \to 0} u_\varepsilon(z) = u(z)$.

To construct such a family of functions, we take an arbitrary infinitely differentiable function $\alpha(t)$ satisfying the conditions $\alpha(t) = 0$ for $t \notin (0,1)$, $\alpha(t) \geqq 0$ for $t \in (0,1)$, and $\int_0^1 t\alpha(t)\, dt = 1/2\pi$. Now set

$$u_\varepsilon(z) = \left(\frac{1}{\varepsilon}\right)^{2n} \left(\frac{i}{2}\right)^n \int\limits_{C^n_{(z')}} u(z + z') \prod_{j=1}^n \alpha\left(\frac{|z_j'|}{\varepsilon}\right) dz_1' \wedge d\bar{z}_1' \wedge \cdots \wedge d\bar{z}_n'. \qquad (2.1.4)$$

It is readily seen that the family of functions $u_\varepsilon(z)$ saitsfies the above conditions. The detailed proof is omitted, since it differs only slightly from the analogous proof for subharmonic functions (see § 1.6 of Chapter 1).

REMARK. *If the function $u(z)$ depends only on the absolute values of its variables, i.e. if throughout some n-circular domain G we have*

$$u(z_1, \cdots, z_n) = u(|z_1|, \cdots, |z_n|),$$

then, as is easily seen from (2.1.4), *the functions $u_\varepsilon(z)$ possess the same property. More precisely, for all sufficiently small $\varepsilon > 0$ the equality*

$$u_\varepsilon(z_1, \cdots, z_n) = u_\varepsilon(|z_1|, \cdots, |z_n|)$$

will hold in any n-circular domain $G_0 \Subset G$.

5. *Differential inequality. Strictly plurisubharmonic functions.* Let $u(z)$ be a twice continuously differentiable plurisubharmonic function in a domain $G \subset C^n$. Then, as follows from the definition, for any $a \in C^n$ and $b \in C^n$ the function $\varphi_{a,b}(w) = u(a_1 w + b_1, \cdots, a_n w + b_n)$ is a subharmonic function of w in the corresponding section of G. Consequently $\Delta_w \varphi_{a,b} \geqq 0$. An elementary calculation shows that

$$\Delta_w \varphi_{a,b} = 4 \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j. \tag{2.1.5}$$

To simplify the notation, we set

$$L(u; z, a) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j.$$

It follows from the above considerations that *for any twice continuously differentiable plurisubharmonic function $u(z)$ the quadratic form $L(u; z, a)$ is positive semidefinite, i.e. the inequality*

$$L(u; z, a) \geqq 0 \tag{2.1.6}$$

holds in the domain for any complex $a_1, \cdots, a_n$.

The converse is also true: *If the form $L(u;z,a)$ is positive semidefinite, the original function $u(z)$ is plurisubharmonic* (provided that $u(z)$ is twice continuously differentiable). This also follows from (2.1.5), since, as we indicated in Chapter 1, § 2, the condition $\Delta_w \varphi \geqq 0$ is not only necessary but also sufficient for the function to be subharmonic.

In some problems of the theory of analytic functions of several variables, importance is attached to functions which satisfy a slightly stronger condition than (2.1.6).

DEFINITION. A twice continuously differentiable function $u(z)$ is said to be *strictly plurisubharmonic* in a domain G if the quadratic form $L(u;z,a)$ is positive definite in G, i.e. if for any $z \in G$

$$\min_{|a|=1} L(u; z, a) > 0.$$

Some elementary properties of strictly plurisubharmonic functions follow.

(a) *Any linear combination* $c_1u_1 + \cdots + c_mu_m$ *of strictly plurisubharmonic functions with nonnegative coefficients* $c_1, \cdots, c_m$ *is again a strictly plurisubharmonic function.*

This assertion is obvious.

(b) *Let* $\varphi(t)$ *be a twice continuously differentiable convex increasing function of the variable* $t \in R^1$. *Let* $u(z)$ *be a strictly plurisubharmonic function. Then the function* $\varphi(u(z))$ *is also strictly plurisubharmonic.*

To prove this it suffices to observe that

$$\Delta_w\{\varphi(u(aw + b))\} = 4\frac{d^2\varphi}{dt^2}\left|\frac{\partial}{\partial w}u(aw + b)\right|^2 + \frac{d\varphi}{dt}L(u; z, a).$$

(c) *If* $u(z)$ *is a strictly plurisubharmonic function, then the approximating functions* $u_\varepsilon(z)$ *defined by* (2.1.4) *are also strictly plurisubharmonic.*

Indeed, by (2.1.4) we have

$$L(u_\varepsilon; z, a) = \left(\frac{i}{2}\right)^n\left(\frac{1}{\varepsilon}\right)^{2n}\int\limits_{C^n_{(z')}} L(u; z + z', a)\prod_{j=1}^{n}\alpha\left(\frac{|z'_j|}{\varepsilon}\right)dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n,$$

and the assertion is now obvious.

We now show that the differentiability condition figuring in the alternative definition of the plurisubharmonic function $u(z)$ by (2.1.6) may be relaxed. When this is done, the derivatives figuring in $L(u;z,a)$ must of course be understood in the distribution-theoretic sense. Indeed, if $u(z)$ is an arbitrary plurisubharmonic function in a domain $G \subset C^n$, then by § 1.4 it can be represented in any domain $G_0 \Subset G$ as a limit $u(z) = \lim_{j\to\infty} u_j(z)$, where $u_j(z)$ is a monotone decreasing sequence of infinitely differentiable plurisubharmonic functions. It follows from the foregoing discussion that $L(u_j; z, a) \geqq 0$ for any j and any $a \in C^n$. Thus for any nonnegative test function $\varphi(z) \in D(G_0)$ we have

$$(L(u_j; z, a), \varphi) = \int\limits_{G_0} L(u_j; z, a)\varphi(z)\, d\omega_z \geqq 0.$$

Integrating by parts, we now get

$$\int\limits_{G_0} L(u_j; z, a)\varphi(z)\, d\omega_z = \int\limits_{G_0} u_j(z)L(\varphi; z, a)\, d\omega_z \geqq 0.$$

Hence, letting $j \to \infty$, we conclude that

$$\int\limits_{G_0} u(z)L(\varphi; z, a)\, d\omega_z \geqq 0,$$

or, what is the same, $(L(u; z, a), \varphi) \geqq 0$. Since the domain $G_0 \Subset G$ is arbitrary, it follows from the last inequality that

$$L(u; z, a) \geqq 0 \tag{2.1.7}$$

in the distribution-theoretic sense in the domain G for any $a \in C^n$.

Now let $u(z) \in D^*(G)$ be a distribution such that inequality (2.1.7) holds in the domain. Setting $a_j = 1$ for $j = 1$ and $a_j = 0$ for $i \neq j$ in this inequality, and summing over i, we get

$$\Delta u = 4 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_i} \geqq 0.$$

Hence by Theorem 1.2.1 it follows that $u(z)$ is a regular distribution, and in fact a subharmonic function. Therefore (see § 1 of Chapter 1) the function $u(z)$ may be represented in any domain $G_0 \Subset G$ as

$$u(z) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2n}} \int_{|z'|<\varepsilon} u(z + z') \alpha\left(\frac{|z'|}{\varepsilon}\right) d\omega_{z'}, \tag{2.1.8}$$

where $\alpha(z)$ is some nonnegative infinitely differentiable function. Moreover, as we have seen, the functions

$$u_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} \int_{|z'|<\varepsilon} u(z + z') \alpha\left(\frac{|z'|}{\varepsilon}\right) d\omega_{z'} \tag{2.1.9}$$

are infinitely differentiable and form a monotone decreasing sequence. We claim that the functions $u_\varepsilon(z)$ are also plurisubharmonic. Indeed, for any function $\varphi(z) \in D(G_0)$, where $G_0 \Subset G$, the following inequality holds for all sufficiently small $\varepsilon > 0$:

$$\begin{aligned}
\int_{G_0} L(u_\varepsilon; z, a)\varphi(z) d\omega_z &= \int_{G_0} u_\varepsilon(z) L(\varphi; z, a) d\omega_z \\
&= \varepsilon^{-2n} \int_{G_0} L(\varphi; z, a) \left\{ \int_{|z'|<\varepsilon} u(z + z') \alpha\left(\frac{|z'|}{\varepsilon}\right) d\omega_{z'} \right\} d\omega_z \\
&= \varepsilon^{-2n} \int_{|z'|<\varepsilon} \alpha\left(\frac{|z'|}{\varepsilon}\right) \left\{ \int_{G_0} u(z + z') L(\varphi; z, a) d\omega_z \right\} d\omega_{z'} \\
&= \varepsilon^{-2n} \int_{|z'|<\varepsilon} \alpha\left(\frac{|z'|}{\varepsilon}\right) (L(u; z, a), \varphi(z - z'))\, d\omega_{z'}.
\end{aligned}$$

Hence, in view of (2.1.7), it follows that

$$\int_{G_0} L(u_\varepsilon; z, a)\varphi(z)\, d\omega_z \geqq 0.$$

Since this inequality holds for any function $\varphi \in D(G_0)$, we also have $L(u_\varepsilon; z, a) \geqq 0$.

Thus the function $u_\varepsilon(z)$ is plurisubharmonic, whence it follows by (2.1.8) that the function $u(z)$ is also plurisubharmonic in any domain $G_0 \Subset G$, and therefore also in the whole domain G. We have thus proved

THEOREM 2.1.1. *A function $u(z) \in D^*(G)$, $G \subset C^n$, is plurisubharmonic if and only if for any $a \in C^n$ the following inequality holds in G:*

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \geqq 0.$$

An analysis of the proof of this theorem shows that we have also incidentally proved the following proposition, mentioned at the beginning of the section:

A function $u(z)$ plurisubharmonic in a domain G is also subharmonic in G.

The converse is false. For example, consider the function $-(|z_1|^2 + |z_2|^2)^{-1}$, which is subharmonic in C^2, and set $z_1 = w$, $z_2 = 1$. Then for $|w| > 1$ we have

$$\Delta_w\left(-\frac{1}{1+|w|^2}\right) = \frac{1-|w|^2}{(1+|w|^2)^3} < 0.$$

Consequently $-(1+|w|^2)^{-1}$ is not subharmonic, and therefore $-(|z_1|^2+|z_2|^2)^{-1}$ is not plurisubharmonic.

We now present two corollaries to Theorem 2.1.1.

COROLLARY 1. *Let $\Phi(z)$ be plurisubharmonic in a domain $G \subset C^n$ containing the surface $S = \{z: z_i = f_i(w), i = 1, \cdots, n, w \in D\}$ where the functions $f_i(w)$ are analytic in a domain $D \subset C^1$. Then $\varphi(w) = \Phi(f_1(w), \cdots, f_n(w))$ is a subharmonic function of the parameter w.*

Indeed,

$$\frac{1}{4}\Delta_w \varphi(w) = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \frac{\partial f_i}{\partial w} \frac{\partial \bar{f}_j}{\partial \bar{w}} = L\left(\Phi; z, \frac{\partial f}{\partial w}\right).$$

Thus $\varphi(w)$ is subharmonic.

COROLLARY 2. *Let $\Phi(z)$ be plurisubharmonic in a domain $G \subset C^n$. Let the functions $f_i(\zeta)$ be analytic in a domain $D \subset C^m$, and such that $f(\zeta) \in G$ for all $\zeta \in D$ (where $f(\zeta) = (f_1(\zeta), \cdots, f_n(\zeta))$. Then the function $\Phi(f(\zeta))$ is plurisubharmonic in D.*

This follows immediately from Corollary 1.

6. *Relation between plurisubharmonic and convex functions.* Plurisubharmonic functions which depend only on the absolute values of the independent variables are often encountered in the theory of analytic functions. In the sequel we shall make repeated use of the following characteristic property of such functions.

THEOREM 2.1.2 (VLADIMIROV [1]). *Let G be a complete n-circular domain in the space C^n, and $\Phi(r) = \Phi(r_1, \cdots, r_n)$ a real-valued function on the set $|G|$.*[3] *Then the function $u(z) = \Phi(|z_1|, \cdots, |z_n|)$ is plurisubharmonic in G if and only if the following four conditions are (simultaneously) satisfied:*

[3] Recall that $|G| = \{r: r_i = |z_i|, i = 1, \cdots, n, z \in G\}$.

(a) *The function* $\Phi(r)$ *is upper semicontinuous on* $|G|$.

(b) $\Phi(r)$ *is convex in the variables* $\ln r_1, \cdots, \ln r_n$ *in the domain* $|G_n|^0$ *consisting of the interior points of the set* $|G|$.

(c) $\Phi(r)$ *is monotone nondecreasing in each of the variables* $r_1, \cdots, r_n$ *on* $|G|$.

(d) *Property* (b) *holds for any* $(n-k)$*-dimensional domain formed by intersecting the set* $|G|$ *with a plane of type* $\{r: r_{i_1} = 0, \cdots, r_{i_k} = 0\}, k = 1, \cdots, n-1$.

PROOF. NECESSITY. Let the function $u(z) = \Phi(|z_1|, \cdots, |z_n|)$ be plurisubharmonic in the domain G. Then it is upper semicontinuous in G, and hence $\Phi(r)$ is upper semicontinuous on $|G|$. Condition (c) follows from the fact that this condition is valid for the function $\mathfrak{M}_u^c(0, r)$, which in this case is simply $\Phi(r)$. To prove (b), we first observe that, by the remark at the end of § 1.4, in every domain $G_0 \Subset G$ the function $u(z)$ is the limit of a monotone decreasing sequence of infinitely differentiable plurisubharmonic functions which depend only on the absolute values of $z_1, \cdots, z_n$. Now, as we remarked in the Introduction, convexity is preserved under passage to a limit. Thus we need prove (b) only for infinitely differentiable functions $u(z)$. But in this case the derivatives $\partial^2 u/\partial z_j \partial \bar{z}_j$ in the quadratic form $L(u;z,a)$ are ordinary derivatives, not derivatives of distributions. Applying the transformation of coordinates $z_j = r_j e^{i\varphi_j}$, $a_j = t_j e^{i\theta_j}$ to this formula, we get

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} = \frac{1}{4}\frac{\partial^2 \Phi}{\partial r_j^2} + \frac{1}{4r_j}\frac{\partial \Phi}{\partial r_j}, \qquad \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\frac{\partial^2 \Phi}{\partial r_j \partial r_k} e^{i(\varphi_k - \varphi_j)} \qquad (j \neq k),$$

and consequently

$$\begin{aligned} L(u; z, a) &= \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial r_j \partial r_k} e^{i(\varphi_k - \varphi_j)} a_j \bar{a}_k + \frac{1}{4}\sum_{j=1}^{n} \frac{\partial \Phi}{\partial r_j}\frac{t_j^2}{r_j} \\ &= \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial r_j \partial r_k} t_j t_k \{\cos(\varphi_j - \theta_j)\cos(\varphi_k - \theta_k) + \sin(\varphi_j - \theta_j)\sin(\varphi_k - \theta_k)\} \\ &\quad + \frac{1}{4}\sum_{j=1}^{n} \frac{\partial \Phi}{\partial r_j}\frac{t_j^2}{r_j}\{\cos^2(\varphi_j - \theta_j) + \sin^2(\varphi_j - \theta_j)\} \qquad (2.10) \\ &= \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial \ln r_j \partial \ln r_k}\frac{t_j t_k}{r_j r_k}\{\cos(\varphi_j - \theta_j)\cos(\varphi_k - \theta_k) \\ &\qquad + \sin(\varphi_j - \theta_j)\sin(\varphi_k - \theta_k)\} \\ &= \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial \ln r_j \partial \ln r_k}\alpha_j \alpha_k + \frac{1}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial \ln r_j \partial \ln r_k}\beta_j \beta_k, \end{aligned}$$

where $\alpha_j = (t_j/r_j)\cos(\varphi_j - \theta_j)$ and $\beta_j = (t_j/r_j)\sin(\varphi_j - \theta_j)$. Since by assumption the function $u(z)$ is plurisubharmonic in G, it follows that $L(u; z, a) \geqq 0$ for $r \in |G|, r_1 \neq 0, \cdots, r_n \neq 0$ and any θ_j and φ_j. Setting $\varphi_k = \theta_k$ or $\varphi_k = \theta_k + \pi$ and recalling that the numbers $t_1, \cdots, t_n$ are arbitrary, we see that for any real $\alpha_1, \cdots, \alpha_n$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial \ln r_j \partial \ln r_k} \alpha_j \alpha_k \geqq 0.$$

Consequently $\Phi(r)$ is convex in $\ln r_1, \cdots, \ln r_n$.

It now remains only to verify condition (d). This follows immediately from (b) and from the fact that for $z_{i_1} = 0, \cdots, z_{i_k} = 0$ the function $u(z)$ is plurisubharmonic in the other variables.

SUFFICIENCY. Let the function $\Phi(r)$ satisfy conditions (a)—(d) of the theorem. Then it follows from (a) that $u(z)$ is upper semicontinuous in the domain G. Further, from condition (b) and the properties of convex functions it follows that $u(z)$ is continuous for $z \in G' = G \setminus \{z: z_1 \cdots z_n = 0\}$. We shall show that $u(z)$ is plurisubharmonic in G'. To prove this, take any domain $D \Subset |G'|$ and a sequence of convex and infinitely differentiable functions in D such that

$$\lim_{j \to \infty} \Phi_j(r) = \Phi(r) \tag{2.1.11}$$

uniformly on D. Set $u_j(z) = \Phi_j(|z_1|, \cdots, |z_n|)$. It follows from (2.1.11) that

$$\lim_{j \to \infty} u_j(z) = u(z)$$

uniformly in the domain $G'' = \{z: (|z_1|, \cdots, |z_n|) \in D\}$, and therefore it will suffice to prove that the functions $u_j(z)$ are plurisubharmonic. This follows from (2.1.10). Indeed, since $\Phi_j(r)$ is convex, both terms in the right-hand side of (2.1.10) are nonnegative. Consequently the inequality $L(u_j; z, a) \geqq 0$ will hold throughout G'' for any $a \in C^n$, and therefore $u_j(z)$ is plurisubharmonic. This proves that $u(z)$ is plurisubharmonic in G'', hence also in G'. Now suppose that, say, $z_{k+1} = \cdots = z_n = 0$, $k > 1$. It follows from condition (d) and the above arguments, applied to the function $u(z_1, \cdots, z_k, 0, \cdots, 0)$, that this function is plurisubharmonic in the intersection of G with the plane $\{z: z_{k+1} = \cdots = z_n = 0\}$, except perhaps for points z such that $z_1 \cdots z_k = 0$. Consequently for any $a \in C^n$ and any sufficiently small $\varepsilon > 0$ the following inequality holds at each point $z^0 = (z_1^0, \cdots, z_k^0, 0, \cdots, 0)$ such that $z_1^0 z_2^0 \cdots z_k^0 \neq 0$:

$$u(z^0) \leqq \frac{1}{2\pi} \int_0^{2\pi} u(z_1^0 + \varepsilon a_1 e^{i\theta}, \cdots, z_k^0 + \varepsilon a_k e^{i\theta}, 0, \cdots, 0)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Phi(|z_1 + \varepsilon a_1 e^{i\theta}|, \cdots, |z_k + \varepsilon a_k e^{i\theta}|, 0, \cdots, 0)\, d\theta.$$

Hence, since the function $\Phi(r_1, \cdots, r_n)$ is monotone nondecreasing in each variable, we conclude that

$$u(z^0) \leqq \frac{1}{2\pi} \int_0^{2\pi} \Phi(|z_1^0 + \varepsilon a_1 e^{i\theta}|, \cdots, |z_k^0 + \varepsilon a_k e^{i\theta}|, |\varepsilon a_{k+1} e^{i\theta}|, \cdots, |\varepsilon a_n e^{i\theta}|)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(z_1^0 + \varepsilon a_1 e^{i\theta}, \cdots, z_k^0 + \varepsilon a_k e^{i\theta}, \varepsilon a_{k+1} e^{i\theta}, \cdots, \varepsilon a_n e^{i\theta})\, d\theta.$$

The validity of the analogous inequality for $z_1^0 = \cdots = z_k^0 = 0$ is obvious. Consequently the subharmanicity of the function

$$\varphi(w) = u(z_1^0 + a_1 w, \cdots, z_k^0 + a_k w, a_{k+1} w, \cdots, a_n w)$$

cannot break down at $w = 0$. This proves that the function $u(z)$ is plurisubharmonic not only in G' but also in G. This completes the proof of the theorem.

COROLLARY 1. *The average* $\mathfrak{M}_u^c(z, r)$ *of a plurisubharmonic function* $u(z)$ *is a convex function of the variables* $\ln r_1, \cdots, \ln r_n$.

Indeed, by the obvious equality

$$\mathfrak{M}_u^c(z; |\zeta_1|, \cdots, |\zeta_n|) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} u(z_1 + \zeta_1 e^{i\theta_1}, \cdots, z_n + \zeta_n e^{i\theta_n})\, d\theta_1 \cdots d\theta_n$$

and property (i) in § 1.2 the function $\mathfrak{M}_u^c(z, |\zeta_1|, \cdots, |\zeta_n|)$ is plurisubharmonic. Hence by Theorem 2.1.2 the function $\mathfrak{M}_u^c(z, r)$ is convex in $\ln r_1, \cdots, \ln r_n$.

COROLLARY 2 (*see* VALIRON [1]). *Let the function* $f(z)$ *be analytic in an n-circular domain* G, *and let*

$$M_f(r) = \max_{\substack{|z_i| = r_i \\ i=1,\cdots,n}} |f(z)|.$$

Then the function $\ln^+ M_f(r)$ *is a convex function of the variables* $\ln r_1, \cdots, \ln r_n$ *in the set* $|G|$.

Indeed, $\ln^+ M_f(|z_1|, \cdots, |z_n|)$ is obviously continuous and admits a representation

$$\ln^+ M_f(|z_1|, \cdots, |z_n|) = \sup_{\substack{-\pi \leqq \theta_i \leqq \pi \\ i=1,\cdots,n}} \ln^+ |f(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n})|.$$

Consequently (see (e) in § 1.2) $\ln^+ M_f(|z_1|, \cdots, |z_n|)$ is plurisubharmonic in G, and hence $\ln^+ M_f(r)$ is a convex function of $\ln r_1, \cdots, \ln r_n$ in G.

The following theorem resembles Theorem 2.1.2 in both content and proof (the latter is omitted).

THEOREM 2.1.3.[4] *A function* $u(x)$, $x \in R^n$, *is convex in a domain* $B \subset R^n$ *if and only if the function* $u(z) \equiv u(x)$, $x + iy = z$, *is plurisubharmonic in the domain* $\{z: x \in B, y \in R^n_{(y)}\}$.

[4] See Vladimirov [1].

To conclude this subsection, we prove an analog of Liouville's Theorem for plurisubharmonic functions.

THEOREM 2.1.4. *If $\Phi(z)$ is a plurisubharmonic function in C^n such that* $\sup_{C^n}\Phi(z) < \infty$, *then* $\Phi(z) \equiv \text{const.}$[5]

PROOF. Set $M = \sup_{C^n}\Phi(z)$. It is obvious that for any $z \in C^n$ and any $r \in R_+^n$ the average $\mathfrak{M}_\Phi^c(z, r)$ is also bounded above by the constant M. Since $\mathfrak{M}_\Phi^c(z, r)$ is convex in $\ln r_1, \cdots, \ln r_n$, it follows immediately that, as a function of r, $\mathfrak{M}_\Phi^c(z,r)$ is identically equal to a constant C, which generally depends on z. Then the average $\mathfrak{N}_\Phi^c(z, r)$ of $\Phi(z)$ over the polydisk $\mathscr{E}(z, r) = \{z' : |z_i' - z_i| < r_i,\ i = 1, \cdots, n\}$ is clearly also equal to the same constant $C = C(z)$ for any r. We claim that in fact $C(z^0) = C(z')$ for any points z^0 and z', i.e. the constant C is also independent of z.

Indeed, setting $r_i' = r_i + |z_i' - z_i^0|$, we see that $\mathscr{E}' = \mathscr{E}(z', r') \supset \mathscr{E} = \mathscr{E}(z^0, r)$, and so

$$C(z') - C(z^0) = \mathfrak{N}_\Phi^c(z', r') - \mathfrak{N}_\Phi^c(z^0, r) = \mathfrak{N}_\Phi^c(z', r') - \frac{1}{\pi^n r_1^2 \cdots r_n^2}\int_{\mathscr{E}} \Phi(z)d\omega_z$$

$$= \mathfrak{N}_\Phi^c(z', r') - \frac{1}{\pi^n r_1^2 \cdots r_n^2}\int_{\mathscr{E}'} \Phi(z)d\omega_{z'} + \frac{1}{\pi^n r_1^2 \cdots r_n^2}\int_{\mathscr{E}'\setminus\mathscr{E}} \Phi(z)d\omega_z$$

$$\leqq \mathfrak{N}_\Phi^c(z', r') - \frac{r_1'^2 \cdots r_n'^2}{r_1^2 \cdots r_n^2}\mathfrak{N}_\Phi^c(z', r') + M\int_{\mathscr{E}'\setminus\mathscr{E}} d\omega_z$$

$$\leqq C(z')\left\{1 - \frac{r_1'^2 \cdots r_n'^2}{r_1^2 \cdots r_n^2}\right\} + M\,\frac{r_1'^2 - r_1^2}{r_1^2}\cdot\frac{r_2'^2 \cdots r_n'^2}{r_2^2 \cdots r_n^2} = o(1),$$

where $o(1) \to 0$ as $r_1 \to \infty, \cdots, r_n \to \infty$. Consequently $C(z') \leqq C(z^0)$. But an analogous argument shows that $C(z^0) \leqq C(z')$, and hence $C(z^0) = C(z')$. Thus $\mathfrak{N}_\Phi^c(z, r) \equiv C$, where C is independent of r and z. Hence it follows immediately (see § 1.3(d)) that $\Phi(z) \equiv C$, and the proof is complete.

7. *Pseudoconvex domains.* Plurisubharmonic functions play an important role in investigations of the positions of singularities of analytic functions of several variables. We present two relevant results, without proofs.[6]

The following two conditions are equivalent.

(a) *There exists a function $u(z)$ plurisubharmonic in a given domain $G \subset C^n$ such that $\{z: u(z) < M\} \Subset G$ for any $M < \infty$.*

(b) *If $\Delta_G(z) = \min_{z' \in \partial G} |z - z'|$, then the function $-\ln \Delta_G(z)$ is plurisubharmonic in the domain G.*

[5] Liouville's Theorem is not valid for subharmonic functions in R^m, $m > 2$; counterexample: $\Phi(x) = -|x|^{\lambda - m}$.

[6] For proofs, the reader may consult, e.g., Hörmander [1].

Domains satisfying either of these conditions are said to be *pseudoconvex*. Their role in the theory of analytic functions of several variables is illustrated by the celebrated theorem of Oka.[7] This theorem may be stated as follows:

A domain $G \subset C^n$ is a domain of holomorphy if and only if it is pseudoconvex.

§ 2. Γ-projection and Γ-capacity

In many problems of analytic function theory and potential theory, sets must be described in terms of their "richness." Ordinary Lebesgue measure turns out to be inadequate for this purpose. Characterization of sets in terms of their capacity is more discriminating. For analytic functions of one variable and subharmonic functions, consideration of capacity often yields good results. However, even this characterization is inadequate for sets arising in the theory of analytic and plurisubharmonic functions of several variables. For example, a disk in C^2 has zero capacity, irrespective of its position in the space, whereas its role for analytic functions of several variables depends essentially on its position. Indeed, it is readily seen that, if the disk lies in the plane $\{z: \operatorname{Im} z_1 = 0, \operatorname{Im} z_2 = 0\}$, then it is a uniqueness set[8] for functions analytic in C^2. This is no longer true if the disk lies in the plane $\{z: z_1 = 0\}$.

In order to achieve a characterization of sets which takes differences of this kind into account, we shall introduce a special projection mapping from the space C^n to a complex space of lower dimension.

1. *Γ-projection.* Let E be a set in C^n. Let $\Delta(E; z_1', \cdots, z_{n-1}')$ denote the intersection of E with the plane $\{z: z_1 = z_1', \cdots, z_{n-1} = z_{n-1}'\}$. We define the *$\Gamma$-projection* of the set $E \subset C^n$, $n > 1$, on the space C^{n-1} to be the set $\Gamma_n^{n-1}(E)$ of all points $(z_1, \cdots, z_{n-1})$ such that

$$\underline{\operatorname{Cap}}_2 \Delta(E; z_1, \cdots, z_{n-1}) > 0.$$

In a certain sense, therefore, the Γ-projection of a set is its "essential" projection.

The Γ-projection of a set $E \subset C^n$, $n > 2$, on the space C^{n-2} is defined to be the set

$$\Gamma_n^{n-2}(E) = \Gamma_{n-1}^{n-2}(\Gamma_n^{n-1}(E)).$$

Similarly we define the sets

$$\Gamma_n^{n-3}(E) = \Gamma_{n-2}^{n-2}(\Gamma_n^{n-2}(E))$$
$$\cdots\cdots\cdots\cdots$$

[7] This theorem was first proved by Oka [1] in 1942 for $n = 2$. Proofs for $n > 2$ were obtained independently in 1953–1954 by Oka [2], Norgeut [1] and Bremermann [1], who also used other equivalent definitions of a pseudoconvex domain.

[8] A set $E \subset C^n$ is said to be a uniqueness set for analytic functions if the condition $f(z) = 0$ $\forall z \in E$ implies that $f(z) \equiv 0$.

$$\Gamma_n^l(E) = \Gamma_{l+1}^l(\Gamma_n^{l+1}(E))$$
$$\cdots\cdots\cdots$$
$$\Gamma_n^1(E) = \Gamma_2^1(\Gamma_n^2(E)).$$

For $E \subset C^1$ we also set $\Gamma_1^1(E) = E$. The set $\Gamma_n^1(E)$ thus defined for all integers $n \geqq 1$, i.e., the Γ-projection of the set E on the space C^1, will be called simply the *Γ-projection* of the set E. Note that $\Gamma_n^1(E) = \Gamma_l^1(\Gamma_n^l(E))$.

REMARK. If E is a Borel set, the set $\Delta(E; z_1, \cdots, z_{n-1})$ is also a Borel set, and therefore capacitable. In this case we may of course write Cap_2 instead of $\underline{\mathrm{Cap}_2}$ in the definition of $\Gamma_n^{n-1}(E)$.

2. *Properties of sets under Γ-projection.* In this subsection we prove a number of lemmas describing certain properties of Γ-projections.

LEMMA 2.2.1. *Let $E \subset C^n$ be a bounded closed set. Then the set $E^\alpha \subset C^{n-1}$ of points $(z_1, \cdots, z_{n-1})$ such that*

$$\mathrm{Cap}_2 \Delta(E; z_1, \cdots, z_{n-1}) \geqq \alpha > 0$$

is also closed.

PROOF. Let $(z_1^{(m)}, \cdots, z_{n-1}^{(m)}) \in E^\alpha$, $m = 1,2,\cdots$, and $(z_1^{(m)}, \cdots, z_{n-1}^{(m)}) \to (z_1^0, \cdots, z_{n-1}^0)$ for $m \to \infty$. Denote

$$\Delta_m = \Delta(E; z_1^{(m)}, \cdots, z_{n-1}^{(m)}) \qquad (m = 1,2,\cdots).$$

Denote the limit set of the sequence $\{\Delta_m\}_1^\infty$ by Δ_0. Thus a point z_n is in the set Δ_0 if and only if it is the limit of a sequence of points $z_n^{(l)}$ $(l = 1,2,\cdots)$ each belonging to some set Δ_{m_l} with $m_l > m_{l-1}$. Since the set E is closed,

$$\Delta_0 \subset \Delta(E; z_1^0, \cdots, z_{n-1}^0).$$

It is easy to see that

$$\Delta_0 = \bigcap_{i=1}^{\infty} \overline{\bigcup_{m=i}^{\infty} \Delta_m}.$$

It follows from the definitions of the sets Δ_m and E^α that $\mathrm{Cap}_2 \Delta_m \geqq \alpha$. Therefore, for any i,

$$\mathrm{Cap}_2 \overline{\bigcup_{m=i}^{\infty} \Delta_m} \geqq \alpha.$$

But then, by (c) in § 4.1 of Chapter 1,

$$\mathrm{Cap}_2 \bigcap_{i=1}^{\infty} \overline{\bigcup_{m=i}^{\infty} \Delta_m} \geqq \alpha,$$

i.e. $\mathrm{Cap}_2 \Delta_0 \geqq \alpha$. Since $\Delta_0 \subset \Delta(E; z_1^0, \cdots, z_{n-1}^0)$,

$$\mathrm{Cap}_2 \Delta(E; z_1^0, \cdots, z_{n-1}^0) \geqq \alpha.$$

This means that $(z_1^0, \cdots, z_{n-1}^0) \in E^\alpha$. Q.E.D.

LEMMA 2.2.2. *If the set $E \subset C^n$ is bounded and closed, then $\Gamma_n^{n-1}(E)$ is an F_σ set.*

PROOF. By definition,

$$\Gamma_n^{n-1}(E) = \{(z_1, \cdots, z_{n-1}) : \mathrm{Cap}_2 \Delta(E; z_1, \cdots, z_{n-1}) > 0\}$$

$$= \bigcup_{j=1}^{\infty} \left\{(z_1, \cdots, z_{n-1}) : \mathrm{Cap}_2 \Delta(E; z_1, \cdots, z_{n-1}) \geqq \frac{1}{j}\right\}.$$

Now each of the sets

$$\{(z_1, \cdots, z_{n-1}) : \mathrm{Cap}_2 \Delta(E; z_1, \cdots, z_{n-1}) \geqq 1/j\}$$

appearing in this union is closed, by Lemma 2.2.1. Theorefore $\Gamma_n^{n-1}(E) \in F_\sigma$. Q.E.D.

LEMMA 2.2.3. *Let the set $E \subset C^n$ be a union $E = \bigcup_1^\infty E_j$, where each E_j is a Borel set. Then*

$$\Gamma_n^{n-1}(E) = \bigcup_{j=1}^{\infty} \Gamma_n^{n-1}(E_j).$$

PROOF. It follows from the assumptions of the lemma that

$$\Delta(E; z_1, \cdots, z_{n-1}) = \bigcup_{j=1}^{\infty} \Delta(E_j; z_1, \cdots, z_{n-1}),$$

where the sets $\Delta(E_j; z_1, \cdots, z_{n-1})$ are Borel sets and therefore capacitable. Consequently the inequality

$$\mathrm{Cap}_2 \Delta(E; z_1, \cdots, z_{n-1}) > 0$$

can hold if and only if, for at least one j,

$$\mathrm{Cap}_2 \Delta(E_j; z_1, \cdots, z_{n-1}) > 0.$$

In other words, $(z_1, \cdots, z_{n-1}) \in \Gamma_n^{n-1}(E)$ if and only if $(z_1, \cdots, z_{n-1}) \in \Gamma_n^{n-1}(E_j)$ for at least one j. Consequently

$$\Gamma_n^{n-1}(E) = \bigcup_{j=1}^{\infty} \Gamma_n^{n-1}(E_j).$$

Q.E.D.

LEMMA 2.2.4. *If $E \subset C^n$ is an F_σ set, then $\Gamma_n^l(E) \in F_\sigma$ for any $l = 1, \cdots, n-1$.*

This is an immediate consequence of Lemmas 2.2.2 and 2.2.3.

LEMMA 2.2.5. *Let $E \subset C^n$ be a closed set such that $\operatorname{Cap}_2 \Gamma_n^1(E) > 0$. Then there exists a system of bounded plane closed sets G_1, $G_2(z_1)$, $\cdots$, $G_n(z_1, \cdots, z_{n-1})$, depending on the indicated parameters, such that the sets generated by the recursion relations*

$$E_1 = G_1$$
$$E_2 = \{(z_1, z_2): z_1 \in E_1, z_2 \in G_2(z_1)\}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$E_n = \{(z_1, \cdots, z_n): (z_1, \cdots, z_{n-1}) \in E_{n-1}, z_n \in G_n(z_1, \cdots, z_{n-1})\}$$

satisfy the following conditions:

(a) $\operatorname{Cap}_2 E_1 > 0$.

(b) $\inf_{E_l} \operatorname{Cap}_2 \Delta(E_{l+1}: z_1, \cdots, z_l) = \alpha_l > 0$ $(l = 1, \cdots, n-1)$.

(c) $E_n \subset E$.

PROOF. The lemma is obviously true for $n = 1$. It will therefore suffice to show that, if it is true for $n = m$, it is also true for $n = m + 1$. Let $n = m + 1$. Without loss of generality we may assume that the set E is bounded. Consider the sets

$$D_j = \{(z_1, \cdots, z_m): \operatorname{Cap}_2 \Delta(E; z_1, \cdots, z_m) \geqq 1/j\}.$$

By Lemma 2.2.1 these sets are closed. It is clear that

$$\Gamma_{m+1}^m(E) = \bigcup_{j=1}^{\infty} D_j,$$

and by Lemma 2.2.3

$$\Gamma_{m+1}^1(E) = \bigcup_{j=1}^{\infty} \Gamma_m^1(D_j).$$

Since $\operatorname{Cap}_2 \Gamma_{m+1}^1(E) > 0$, while $\Gamma_m^1(D_j)$ are F_σ sets and therefore capacitable, it follows that, for at least one index $j = j_0$,

$$\operatorname{Cap}_2 \Gamma_m^1(D_{j_0}) > 0. \tag{2.2.1}$$

By the induction hypothesis (for $n = m$) it follows from (2.2.1) that there exists a system of closed bounded sets G_1, $G_2(z_1)$, $\cdots$, $G_m(z_1, \cdots, z_{m-1})$ such that the corresponding sets $E_1, \cdots, E_m$ satisfy conditions (a), (b) and the condition $E_m \subset D_{j_0}$. Now set

$$G_{m+1}(z_1, \cdots, z_m) = \Delta(E; z_1, \cdots, z_m)$$

and

$$E_{m+1} = \{(z_1, \cdots, z_{m+1}) : (z_1, \cdots, z_m) \in E_m,\ z_{m+1} \in G_{m+1}(z_1, \cdots, z_m)\}.$$

It is obvious that the system $G_1, \cdots, G_{m+1}$ satisfies all conditions of the lemma for $n = m + 1$. This completes the proof of Lemma 2.2.5.

Before proceeding to Lemmas 2.2.6 and 2.2.7 which concern the Lebesgue measure of sets with Γ-projections of zero capacity, we introduce the notation $\mu_l(N)$ for the l-dimensional Lebesgue measure of a set $N \subset R^l$.

LEMMA 2.2.6. *Let $E \subset C^n$ be a Borel set such that*

$$\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) = 0.$$

Then $\mu_{2n}(E) = 0$.

PROOF. Let E_1 denote the set of points $(z_1', \cdots, z_{n-1}')$ such that

$$\mu_2(\Delta(E; z_1', \cdots, z_{n-1}')) > 0.$$

It is clear that $E_1 \subset \Gamma_n^{n-1}(E)$. Suppose that $\mu_{2n}(E) > 0$. Then it follows from Fubini's Theorem that $\mu_{2n-2}(E_1) > 0$. Repeating this argument with the set E replaced by E_1, we get a set E_2 such that $E_2 \subset \Gamma_n^{n-2}(E)$ and $\mu_{2n-4}(E_2) > 0$. Repeating the procedure $n - 2$ times, we get a set $E_{n-1} \subset \Gamma_n^1(E)$ such that $\mu_2(E_{n-1}) > 0$. Hence it follows immediately that $\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) > 0$, contradicting the assumption. Consequently the inequality $\mu_{2n}(E) > 0$ is impossible. Q.E.D.

LEMMA 2.2.7. *Let $E \subset C^n$ be a Borel set such that*

$$\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) = 0.$$

Then the set $E' = E \cap \{z : \mathrm{Im}\, z_1 = 0, \cdots, \mathrm{Im}\, z_n = 0\}$ satisfies the condition $\mu_n(E') = 0$.

PROOF. Suppose that $\mu_n(E') > 0$, and consider the set E_1' of all points $(x_1', \cdots, x_{n-1}')$ such that

$$\mu_1(E' \cap \{x : x \in R^n, x = x_1', \cdots, x_{n-1} = x_{n-1}'\}) > 0.$$

By Fubini's Theorem, the set E_1' must satisfy the condition $\mu_{n-1}(E_1') > 0$. On the other hand, since the intersection of an arbitrary Borel set $N \subset R^2$ of zero capacity with any straight line has linear Lebesgue measure zero, it follows that $E_1' \subset \Gamma_n^{n-1}(E)$. As in the proof of the preceding lemma, we repeat this argument $n - 2$ times, substituting the new set for the old at each step. We finally arrive at a linear set $E_{n-1}' \subset \Gamma_n^1(E)$ such that $\mu_1(E_{n-1}') > 0$. But then $\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) > 0$, a contradiction. Thus the inequality $\mu_n(E') > 0$ is impossible, and the proof is complete.

3. *Γ-capacity.* A good numerical index of the "richness" of sets in C^n is the inner capacity (for capacitable sets, capacity) of their Γ-projections. The "rich" sets will be those for which $\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) > 0$, while the "lean" sets are characterized

by $\underline{\mathrm{Cap}}_2\,\Gamma^1_\pi(E) = 0$. As a reflection of the nature of the objects we are studying in C^n, this characterization proves to be more faithful than the capacity of sets. Thus, for the example considered at the beginning of this section, the capacity of the Γ-projection of a disk in the plane $\mathrm{Im}\, z_1 = 0$, $\mathrm{Im}\, z_2 = 0$ is positive, while the capacity of the Γ-projection of the same disk in the plane $z_1 = 0$ is zero. Nevertheless, characterization of sets in terms of the capacity of their Γ-projections has certain disadvantages. One can construct sets for which the capacity of the Γ-projection depends essentially on the order of the coordinates. The following examples of such sets are due to V. Šeinov.

EXAMPLE 1. In the space C^2 consider the set

$$E = \{(z_1, z_2)\colon |z_1| + |z_2| = 1,\ \mathrm{Im}\, z_1 = 0\}.$$

For $-1 < x_1' < 1$ we have

$$\Delta(E; x_1') = \{z_2\colon z_2 = (1 - |x_1'|)e^{i\theta},\ 0 \leqq \theta \leqq 2\pi\}.$$

Consequently $\mathrm{Cap}_2\,\Delta(E; x_1') > 0$ and $x_1' \in \Gamma^1_2(E)$, whence it follows that $\mathrm{Cap}_2\,\Gamma^1_2(E) > 0$. But if we interchange the variables z_1 and z_2, i.e. consider instead of E the set

$$E' = \{(z_1, z_2)\colon |z_1| + |z_2| = 1,\ \mathrm{Im}\, z_2 = 0\},$$

then for any z_1 in the disk $|z_1| \leqq 1$ we have

$$\Delta(E'; z_1) = \pm\,(1 - |z_1|),$$

while for $|z_1| > 1$ we have $\Delta(E'; z_1) = \varnothing$. Consequently

$$\mathrm{Cap}_2\,\Delta(E'; z_1) = 0 \quad \forall z_1 \in C^1, \qquad \text{and} \qquad \Gamma^1_2(E') = \varnothing.$$

EXAMPLE 2. In C^3, consider the set

$$E = \{(z_1, z_2, z_3)\colon |z_1| < 1,\ |z_2| + |z_3| = 1,\ \mathrm{Im}\, z_3 = 0\}.$$

In view of the properties of the set in Example 1, we see that $\Gamma^1_3(E) = \{z_1\colon |z_1| \leqq 1\}$, and hence $\mathrm{Cap}_2\,\Gamma^1_3(E) > 0$. But if we interchange the variables z_2 and z_3, then, as is readily seen, $\Gamma^1_3(E) = \varnothing$, and this set therefore has zero capacity.

A characterization of sets without these disadvantages, i.e. invariant under permutations of the coordinates and, even more, invariant under a certain class of linear transformations, is obtained as follows.

DEFINITION. The *Γ-capacity* of a set $E \subset C^n$ is the quantity

$$\Gamma\text{-Cap}\,E = \sup_{\alpha \in T} \underline{\mathrm{Cap}}_2\,\Gamma^1_\pi(\alpha E),^{9)}$$

---

9) Γ-capacity can be defined somewhat differently:

$$\Gamma\text{-Cap}\,E = \sup_{\alpha \in T} \overline{\mathrm{Cap}}_2\,\Gamma^1_\pi(\alpha E).$$

It follows from the lemmas in § 2.2 that these two definitions are equivalent for closed sets E. We do not know whether they are equivalent for arbitrary Borel sets.

where T is the class of unitary transformations of the space C^n.[10]

It is obvious that the quantity Γ-Cap E thus defined is invariant under unitary transformations of the space C^n.

The following criterion for the vanishing of Γ-capacity is extremely useful.

LEMMA 2.2.8. *Let $E \subset C^n$, $E \neq C^n$, and assume that the intersection of E with any analytic plane $\{z: z_i = a_i w + b_i, i = 1, \cdots, n, |w| < \infty\}$ which is not a subset of E has inner capacity zero. Then the Γ-capacity of E is zero.*

PROOF. Since the conditions imposed on the set E are invariant under unitary transformations of C^n, it will suffice to prove that

$$\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) = 0.$$

Suppose that this is false, i.e.

$$\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) > 0.$$

Consider any point $(z_1', \cdots, z_{n-1}') \in \Gamma_n^{n-1}(E)$. It follows from the assumptions of the lemma and the definition of the set $\Gamma_n^{n-1}(E)$ that the entire plane $\{z: z_1 = z_1', \cdots, z_{n-1} = z_{n-1}'\}$ lies in the set E. Therefore

$$\{z: (z_1, \cdots, z_{n-1}) \in \Gamma_n^{n-1}(E), |z_n| < \infty\} \subset E, \tag{2.2.2}$$

and hence any complex one-dimensional analytic plane in C^{n-1} spanned by the variables $z_1, \cdots, z_{n-1}$ either is a subset of $\Gamma_n^{n-1}(E)$ or its intersection with $\Gamma_n^{n-1}(E)$ has inner capacity zero. Since $E \neq C^n$, it follows from (2.2.2) that $\Gamma_n^{n-1}(E) \neq C^{n-1}$. Consequently the set $\Gamma_n^{n-1}(E)$ satisfies the assumptions of the lemma in the space C^{n-1}. Repetition of this argument leads to the conclusion that the set $\Gamma_n^1(E)$ also satisfies the assumptions of the lemma in the space C^1. Thus the assumption

$$\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) > 0$$

implies that $\Gamma_n^1(E) = C^1$, which contradicts the assumption that the set in question does not coincide with the entire space. Consequently the inequality $\underline{\mathrm{Cap}}_2\, \Gamma_n^1(E) > 0$ cannot hold. Q.E.D.

In our oft-cited example of the disk in C^n, we saw that the Γ-projection of a disk lying in the analytic plane $z_1 = 0$ has zero capacity. It can be shown that, in general, the Γ-capacity of a disk in an arbitrary analytic plane is zero. Moreover, we have

[10] A unitary transformation of the space is a transformation $z \to \zeta$, $\zeta_j = \langle z, a^{(j)} \rangle$, $j = 1, \cdots, n$, where the vectors $a^{(j)} \in C^n$ satisfy the condition $|z| = |\zeta|$ for all $\zeta \in C^n$.

LEMMA 2.2.9. *The Γ-capacity of any analytic set*[11] $E \subset C^n$ *is zero.*

PROOF. It will clearly suffice to prove that the capacity of the Γ-projection of any set of the form

$$E' = \{(z: z \in \mathscr{E}(z_0, r), f(z) = 0\},$$

where $f(z)$ is analytic in the polydisk $\mathscr{E}(z_0, r)$, is zero. Note that, as follows from the theory of analytic sets,[12] the intersection of E' with any complex one-dimensional analytic plane is either at most countable or identical with the intersection of the plane with the polydisk $\mathscr{E}(z_0, r)$. Therefore the same arguments that led in Lemma 2.2.8 to the equality $\underline{\mathrm{Cap}}_2 \Gamma_n^1(E) = 0$ imply here that $\mathrm{Cap}_2 \Gamma_n^1(E') = 0$. Consequently Γ-Cap $E = 0$. Q.E.D.

We end this section with the remark that it is not known whether the equality Γ-Cap $E = 0$, where E is a compact set in C^n, implies that $\mathrm{Cap}_{2n} E = 0$.

§ 3. The sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$ for plurisubharmonic functions

Let $\mathfrak{F} = \{u_\alpha(z)\}$ be a family of plurisubharmonic functions, uniformly bounded above in a domain $G \subset C^n$. Let

$$u(z) = u_{\mathfrak{F}}(z) = \sup_{\mathfrak{F}} u_\alpha(z),$$

and let $u^*(z)$ be the regularization of $u(z)$. It is easily seen that *the function $u^*(z)$ is plurisubharmonic.* Indeed, the function

$$\varphi_{z',\alpha}(w) = u_\alpha(aw + b + z') \qquad (a, b \in C^n)$$

is a subharmonic function of the complex scalar w. Consequently for sufficiently small $r > 0$ we have

$$\varphi_{z',\alpha}(w) \leqq \frac{1}{2\pi} \int_0^{2\pi} \varphi_{z',\alpha}(w + re^{i\theta})\, d\theta.$$

This inequality obviously remains valid when we form the least upper bound over α and $z' \in \{z': |z'| < \varepsilon\}$, and also when we then let $\varepsilon \to 0$. Thus the function $\psi(w) = u^*(aw + b)$ satisfies the inequality

$$\psi(w) \leqq \frac{1}{2\pi} \int_0^{2\pi} \psi(w + re^{i\theta})\, d\theta.$$

Now observe that the function $u^*(z)$ is upper semicontinuous, and hence so is $\psi(w)$. Consequently $\psi(w)$ is subharmonic, and so $u^*(z)$ is plurisubharmonic.

---

[11] *A* set $E \subset C^n$ is said to be analytic if it is defined in the neighborhood of any of its points by a system of functions $f_1(z), \cdots, f_N(z)$ analytic in C^n:

$$E = \{z: f_1(z) = 0, \cdots, f_N(z) = 0\}.$$

[12] See, for example, Hervé [1].

Similarly one proves that if the index α takes on real values arbitrarily close to $+\infty$, *the regularization* $v^*(z)$ *of the function*

$$v(z) = v_{\mathfrak{F}}(z) = \overline{\lim_{\alpha \to \infty}} u_\alpha(z)$$

is also a plurisubharmonic function. Here, as in § 5 of Chapter 1, we wish to characterize the "richness" of the sets

$$T_{\mathfrak{F}} = \{z: u(z) < u^*(z)\} \qquad \text{and} \qquad S_{\mathfrak{F}} = \{z: v(z) < v^*(z)\}.$$

Theorems 1.5.1 and 1.5.2 are of course valid in this case. However, since the class of plurisubharmonic functions is smaller than the class of subharmonic functions, it is natural to look for a more delicate characterization. One result of this type can be phrased in terms of Γ-capacity, as the following theorem shows.

THEOREM 2.3.1 (RONKIN [7]). *Let* $\mathfrak{F} = \{u_\alpha(z)\}$ *be a family of functions, plurisubharmonic and uniformly bounded above in a domain* $G \subset C^n$. *Then the set* $T_{\mathfrak{F},\varepsilon}$ *of points* $z \in G$ *at which the function* $u(z) = \sup_\alpha u_\alpha(z)$ *and its regularization* $u^*(z)$ *satisfy the inequality* $u(z) \leqq u^*(z) - \varepsilon$ *is such that*

$$\Gamma\text{-Cap } T_{\mathfrak{F},\varepsilon} = 0.$$

The proof is based on the following lemmas.

LEMMA 2.3.1. *Let the functions* $\varphi(z)$ *and* $\Phi(z)$ *be subharmonic in a domain* $G \subset C^n$. *Assume that* $\varphi(z_0) \leqq \Phi(z_0) - \varepsilon$ *for some point* $z_0 \in G$ *and* $\varepsilon > 0$. *Then any arbitrarily small neighborhood of* z_0 *contains a closed set, of positive Lebesgue measure, at any point of which* $\varphi(z) \leqq \Phi(z) - \varepsilon/2$.

PROOF. Suppose the assertion false. Then, in an arbitrarily small neighborhood of the point z_0, the set of points z at which $\varphi(z) \leqq \Phi(z) - \varepsilon/2$ has Lebesgue measure zero. Thus the inequality $\varphi(z) \geqq \Phi(z) - \varepsilon/2$ holds almost everywhere in these neighborhoods: Averaging both sides of this inequality over a ball, we see that for arbitrarily small $r > 0$

$$\mathfrak{N}_\varphi(z_0, r) \geqq \mathfrak{N}_\Phi(z_0, r) - \varepsilon/2.$$

Letting $r \to 0$, we conclude that $\varphi(z_0) \geqq \Phi(z_0) - \varepsilon/2$, and this contradicts the assumption of the lemma. Q.E.D.

LEMMA 2.3.2. *Let* E *be a Borel set of points* $(z, z_{n+1}) \in C^{n+1}$, *and* E_1 *a Borel set of positive capacity in the* z_{n+1}*-plane. Assume that for any* $z^0_{n+1} \in E_1$ *the n-dimensional Lebesgue measure of the intersection*

$$E \cap \{(z, z_{n+1}): z_{n+1} = z^0_{n+1}\}$$

is positive. Then there exists a set of positive Lebesgue measure in C^n *such that*

$\operatorname{Cap}_2 \Delta(E; z) > 0$ *for any* $z \in D$ (*for the notation see* §2).

PROOF. We first observe that without loss of generality the set E_1 may be assumed compact. Let ν be the Robin distribution on E_1. As we pointed out in Chapter 1, § 4, the Robin distribution of a compact set of positive capacity exists and possesses the following important property:

If $E_1' \subset E$ *is a Borel set such that* $\operatorname{Cap} E_1' = 0$, *then* $\nu(E_1') = 0$.

Let E^* denote the set of points $(z, z_{n+1}) \in E$ such that $z_{n+1} \in E_1$. It is obvious that E^* is a Borel set.

Consider the integral

$$\int_{C^{n+1}} \chi(z, z_{n+1})\, d\omega_z\, d\nu_{z_{n+1}},$$

where $d\omega_z$ is the volume element in the space C^n and $\chi(z, z_{n+1})$ the characteristic function of the set E^*.

By Fubini's Theorem[13)]

$$\int_{C^{n+1}} \chi(z, z_{n+1})\, d\omega_z\, d\nu_{z_{n+1}} = \int_{E_1} d\nu_{z_{n+1}} \int_{C^n} \chi(z, z_{n+1})\, d\omega_z,$$

and, by the properties of the set E_1 indicated in the statement of the lemma,

$$\int_{E_1} d\nu_{z_{n+1}} \int_{C^n} \chi(z, z_{n+1})\, d\omega_z > 0.$$

Changing the order of integration in this integral, we get

$$\int_{C^n} d\omega_z \int_{E_1} \chi(z, z_{n+1})\, d\nu_{z_{n+1}} > 0.$$

Consequently the set D of points z such that $\int_{E_1} \chi(z, z_{n+1})\, d\nu_{z_{n+1}} > 0$ has positive Lebesgue measure. Now

$$\int_{E_1} \chi(z, z_{n+1})\, d\nu_{z_{n+1}} = \nu(\Delta(E^*; z)).$$

Therefore $\nu(\Delta(E^*; z)) > 0$ for any $z \in D$. Moreover, the above property of the Robin distribution implies that $\operatorname{Cap}_2 \Delta(E^*; z) > 0$ for all $z \in D$, and, since $E^* \subset E$, also $\operatorname{Cap}_2 \Delta(E; z) > 0$ for all $z \in D$. Consequently D is the required set. Q.E.D.

Before stating the next lemma, we introduce the notation $u^*(z_1, \cdots, z_q, \lceil z_{q+1}, \cdots, z_n)$ for the regularization of the function $u(z_1, \cdots, z_n)$ with respect to the variables following the symbol $\lceil$, for fixed values of the other variables. Similarly, $u(z_1, \cdots, z_q \rceil, z_{q+1}, \cdots, z_n)$ will denote the regularization of the function $u(z_1, \cdots, z_n)$ with respect to the variables preceding the symbol $\rceil$.

LEMMA 2.3.3. *Let the functions* $u_\infty(z)$, $u(z)$ *and* $u^*(z)$ *be the same as in Theorem* 2.3.1. *Let* $\Phi(z)$ *be a function plurisubharmonic in the domain* G, *such that* $u(z) \leqq \Phi(z)$

[13)] See, for example, Saks [1].

for all $z \in G$. Assume that for some $\varepsilon > 0$, for every point z_1' in a Borel set $S \subset C^1$ of positive capacity, there are points $z_2', \cdots, z_n'$ such that

$$u^*(z_1', \lceil z_2', \cdots, z_n') \leqq \Phi(z_1', \cdots, z_n') - \varepsilon.$$

Then the inequality $u^*(z) \leqq \Phi(z) - \varepsilon/100$ holds at some point of G.

PROOF. As was shown in the proof of Lemma 1.5.5, any family $\mathfrak{F}$ of functions $u_\alpha(x)$, plurisubharmonic and uniformly bounded in G, contains a countable subfamily $\mathfrak{F}_0$ such that $u_{\mathfrak{F}}^*(z) = u_{\mathfrak{F}_0}^*(z)$. We may therefore assume without loss of generality that the family $\mathfrak{F}$ in the statement of our lemma is countable. Then, by Lemma 1.5.1, the set E_ε of all points z at which $u(z) \leqq \Phi(z) - \varepsilon$ is a Borel set. Let us apply Lemma 2.3.1 to the functions $u^*(z_1, \lceil z_2, \cdots, z_n)$ and $\Phi(z)$, which are subharmonic in the variables $x_2 = \operatorname{Re} z_2$, $y_2 = \operatorname{Im} z_2, \cdots, x_n = \operatorname{Re} z_n$, $y_n = \operatorname{Im} z_n$. The result is that to every point $z_1 \in S$ there corresponds a set $T = T(z_1)$ in the space of the variables $z_2, \cdots, z_n$ such that $\mu_{2n-2}(T) > 0$,[14] and such that

$$u^*(z_1, \lceil z_2, \cdots, z_n) \leqq \Phi(z) - \varepsilon/2$$

for $(z_2, \cdots, z_n) \in T(z_1)$. Obviously

$$\{z: z_1 \in S, (z_2, \cdots, z_n) \in T(z_1)\} \subset E_{\varepsilon/2}.$$

Thus the sets $E_{\varepsilon/2}$ and S satisfy the assumptions of Lemma 2.3.2. It follows that the space of $z_2, \cdots, z_n$ contains a set T' such that $\mu_{2n-2}(T') > 0$ and for $(z_2', \cdots, z_n') \in T'$ the set $\{z_1: (z_1, z_2', \cdots, z_n') \in E_{\varepsilon/2}\}$ has positive capacity. In other words, the set of points z_1 such that, for $(z_2', \cdots, z_n') \in T'$,

$$u(z_1, z_2', \cdots, z_n') \leqq \Phi(z_1, z_2', \cdots, z_n') - \varepsilon/2$$

has positive measure. Treating the functions $u_\alpha(z_1, z_2', \cdots, z_n')$ and $\Phi(z_1, z_2', \cdots, z_n')$ as subharmonic functions of the single variable z_1 for fixed $z_2', \cdots, z_n'$, let us apply Lemma 1.5.7. It follows that the regularization of the function $u(z_1, z_2', \cdots, z_n')$ with respect to z_1 satisfies the following inequality for at least one point $z_1 = z_1^0$:

$$u^*(z_1^0 \rceil, z_2', \cdots, z_n') \leqq \Phi(z_1^0, z_2', \cdots, z_n') - \varepsilon/10.$$

Again applying Lemma 2.3.1, we see that for $(z_2', \cdots, z_n') \in T'$ the inequality

$$u^*(z_1 \rceil, z_2', \cdots, z_n') \leqq \Phi(z_1, z_2', \cdots, z_n') - \varepsilon/20$$

holds on some set $K = K(z_2', \cdots, z_n') \subset C^1_{(z_1)}$ of positive Lebesgue measure $\mu_2(K)$. A fortiori the inequality

$$u(z_1, z_2', \cdots, z_n') \leqq \Phi(z_1, z_2', \cdots, z_n') - \varepsilon/20$$

holds on the same set. Consequently the intersection of the set $E_{\varepsilon/20}$ with $\{z: z_2 =$

[14] As before, $\mu_l(D)$, $D \subset R^l$, denotes the l-dimensional Lebesgue measure of the set D.

$z_2', \cdots, z_n = z_n'\}$ for $(z_2', \cdots, z_n') \in T'$ has positive plane Lebesgue measure. Hence, since $\mu_{2n-2}(T') > 0$, it follows that $\mu_{2n}(E_{\varepsilon/20}) > 0$. We again apply Lemma 1.5.7, treating $u_\alpha(z)$ and $\Phi(z)$ as subharmonic functions in the domain G. Thus $u^*(z) \leqq \Phi(z) - \varepsilon/100$ for at least one point $z = z^0$ in G. This completes the proof of Lemma 2.3.3.

We now prove Theorem 2.3.1. Note first that both the condition of the theorem and its statement are invariant under linear transformations of the space C^n. It will therefore suffice to show that

$$\underline{\mathrm{Cap}}_2 \Gamma_n^1(T_{\mathfrak{F},\varepsilon}) = 0.$$

Moreover, as in Lemma 1.5.3 there is no loss of generality in assuming the family $\mathfrak{F}$ to be countable and the set $T_{\mathfrak{F},\varepsilon}$ to be a Borel set.

Suppose that

$$\underline{\mathrm{Cap}}_2 \Gamma_n^1(T_{\mathfrak{F},\varepsilon}) > 0. \tag{2.3.1}$$

Then the Γ-projection $\Gamma_n^l(T_{\mathfrak{F},\varepsilon})$ of the set $T_{\mathfrak{F},\varepsilon}$ in the space C^l is not empty for any $l = 1, \cdots, n-1$. Let $(z_1', \cdots, z_{n-1}') \in \Gamma_n^{n-1}(T_{\mathfrak{F},\varepsilon})$. Then, by the definition of $\Gamma_n^{n-1}(T_{\mathfrak{F},\varepsilon})$, the inequality

$$u(z_1', \cdots, z_{n-1}', z_n) \leqq u^*(z_1', \cdots, z_{n-1}', z_n) - \varepsilon$$

holds on some set of positive capacity in the z_n-plane. Treating the functions $u_\alpha(z_1', \cdots, z_{n-1}', z_n)$ and $u^*(z_1', \cdots, z_{n-1}', z_n)$ as functions of the single variable z_n, we see from Lemma 1.5.7 that for some $z_n = z_n^{(1)}$

$$u^*(z_1', \cdots, z_{n-1}', \Gamma z_n^{(1)}) \leqq u^*(z_1', \cdots, z_{n-1}', z_n^{(1)}) - \varepsilon/5. \tag{2.3.2}$$

Here, as in Lemma 2.3.3, the symbol Γ means that the regularization is taken over the variables following the symbol.

Now let $(z_1', \cdots, z_{n-2}') \in \Gamma_n^{n-2}(T_{\mathfrak{F},\varepsilon})$. Then, by the definition of the Γ-projection, the set

$$\Delta(\Gamma_n^{n-1}(T_{\mathfrak{F},\varepsilon}); z_1', \cdots z_{n-2}')$$

$$= \Gamma_n^{n-1}(T_{\mathfrak{F},\varepsilon}) \cap \{(z_1, \cdots, z_{n-1}) : z_1 = z_1', \cdots, z_{n-2} = z_{n-2}'\}$$

has positive capacity. Hence by (2.3.2) it follows that the set

$$T_{\mathfrak{F},\varepsilon/5} \cap \{z : z_1 = z_1', \cdots, z_{n-2} = z_{n-2}'\},$$

considered in the space of z_{n-1}, z_n, satisfies the assumptions of Lemma 2.3.3 with $\Phi(z) = u^*(z)$. Therefore for $(z_1', \cdots, z_{n-2}') \in \Gamma_n^{n-2}(T_{\mathfrak{F},\varepsilon})$ there exists a point $(z_{n-1}^{(2)}, z_n^{(2)})$ such that

$$u^*(z_1', \cdots, z_{n-2}', \Gamma z_{n-1}^{(2)}, z_n^{(2)} < u^*(z_1', \cdots, z_{n-2}', z_{n-1}^{(2)}, z_n^{(2)}) - \varepsilon/5 \cdot 100.$$

Note that we have not yet used assumption (2.3.1), but only the fact that the sets $\Gamma_n^{n-1}(T_{\mathfrak{F},\varepsilon})$ and $\Gamma_n^{n-2}(T_{\mathfrak{F},\varepsilon})$ are nonempty.

We now repeat this reasoning $n-2$ times, applying Lemma 2.3.3 successively to the corresponding sets in the spaces $C^3, \cdots, C^n$. The last step of this procedure involves a direct application of condition (2.3.1). The final result is that there exists a point $z^{(n)} = (z_1^{(n)}, \cdots, z_n^{(n)})$ such that

$$u^*(z^{(n)}) \leqq u^*(z^{(n)}) - \varepsilon/5 \cdot 10^{2n-2},$$

and this is impossible. Consequently

$$\underline{\mathrm{Cap}}_2 \Gamma_n^1(T_{\mathfrak{F},\varepsilon}) = 0.$$

Q.E.D.

THEOREM 2.3.2 (RONKIN [7]). *Let $\mathfrak{F}$ be a family of functions $u_\alpha(z)$, plurisubharmonic and uniformly bounded above in a domain $G \subset C^n$, such that the functions* $\exp\{u_\alpha(z)\}$ *are continuous in G.*[15] *Then the set $T_{\mathfrak{F}}$ of all points $z \in G$ at which the function $u(z) = \sup_{\mathfrak{F}} u_\alpha(z)$ is strictly smaller than its regularization $u^*(z)$ satisfies the condition Γ-Cap $T_{\mathfrak{F}} = 0$.*

PROOF. It is obvious that $T_{\mathfrak{F}} = \bigcup_{\varepsilon>0} T_{\mathfrak{F},\varepsilon}$, where the sets $T_{\mathfrak{F},\varepsilon}$ are the same as in Theorem 2.3.1. It is also obvious that

$$T_{\mathfrak{F}} = \bigcup_{j=1}^{\infty} T_{\mathfrak{F},1/j}.$$

By Lemma 1.5.2, $T_{\mathfrak{F},1/j}$ is an F_σ set. Therefore (see Lemmas 2.2.3 and 2.2.4)

$$\Gamma_n^1(T_{\mathfrak{F}}) = \bigcup_{j=1}^{\infty} \Gamma_n^1(T_{\mathfrak{F},1/j})$$

and $\Gamma_n^1(T_{\mathfrak{F},1/j}) \in F_\sigma$ for any $j = 1, 2, \cdots$. Hence, by Theorem 2.3.1, $\mathrm{Cap}_2 \Gamma_n^1(T_{\mathfrak{F}}) = 0$, and so Γ-Cap $T_{\mathfrak{F}} = 0$. Q.E.D.

THEOREM 2.3.3. *Let $\mathfrak{F} = \{u_\alpha(z)\}_{\alpha \in (0,\infty)}$ be a family of functions, plurisubharmonic and uniformly bounded in a domain $G \subset C^n$, such that the functions* $\exp\{u_\alpha(z)\}$ *are continuous in G. Then the set $S_{\mathfrak{F}}$ of all points $z \in G$ at which the function*

$$v(z) = \overline{\lim_{\alpha \to \infty}} u_\alpha(z)$$

is strictly smaller than its regularization $v^*(z)$ satisfies the condition

$$\Gamma\text{-Cap } S_{\mathfrak{F}} = 0.$$

[15] This condition always holds for families of plurisubharmonic functions arising in investigations of entire functions.

The proof is omitted, since it is entirely similar to that of Theorem 1.5.2.

Lelong [6] has given another characterization of the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$. He proves that the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$ are unions $\bigcup_1^n E_j$, where each set E_j satisfies the condition

$$\operatorname{Cap}_2 \left\{ E_j \bigcap \left\{ z : \underset{i \neq j}{z_i = z_i'} \right\} \right\} = 0$$

for arbitrary $z_1', \cdots, z_{j-1}', z_{j+1}', \cdots, z_n'$.

When the functions $e^{u_\alpha(z)}$ are continuous, this statement follows, as is readily seen, from Theorems 2.3.2 and 2.3.3.

The question whether Theorems 2.3.2 and 2.3.3 provide a complete characterization of the sets $T_{\mathfrak{F}}$ and $S_{\mathfrak{F}}$ is open.

§ 4. Representation of functions plurisubharmonic throughout C^n

As we remarked in § 1 of this chapter, plurisubharmonic functions are also subharmonic. They therefore have Riesz decomposition (Theorem 1.2.1′)

$$u(z) = \int_{C^n} h(\zeta - z)\, d\mu_\zeta + \Phi_{C^n}(z),$$

where $h(z) = -|z|^{2-2n}$, μ is the Riesz measure associated with the function $u(z)$, and $\Phi_{C^n}(z)$ is a harmonic function. This decomposition is seldom used in the theory of plurisubharmonic functions, since in general neither of its terms are plurisubharmonic. This disadvantage is absent in the representation given in § 3 of Chapter 1 for functions subharmonic in the entire space. To be precise:

THEOREM 2.4.1 (LELONG [3], [7]). *Let $u(z)$ be a function, plurisubharmonic in C^n, such that the Riesz associated measure μ has finite convergence exponent and, for some value of $t = t_1 > 0$, satisfies the condition $\mu(t_1) = 0$, where $\mu(t) = \int_{|z| \leqq t} d\mu$. Then $u(z)$ admits the representation $u(z) = \Phi(z) + J_\mu(z)$, where $\Phi(z)$ is a pluriharmonic function*[16] *and $J_\mu(z)$ is the canonical potential of the measure μ.*[17]

PROOF. We first prove the following lemmas.

LEMMA 2.4.1. *If the function $u(z)$ is both harmonic and plurisubharmonic in a domain $G \subset C^n$, then it is also pluriharmonic in G.*

PROOF. Since $u(z)$ is plurisubharmonic in G, the following inequality will hold in G for any complex $\lambda_1, \cdots, \lambda_n$:

---

[16] Recall that a differentiable function $u(z)$ is said to be pluriharmonic in a domain $G \subset C^n$ if in this domain

$$\partial^2 u / \partial z_p \partial \bar{z}_q = 0 \qquad (p, q = 1, \cdots, n).$$

[17] Concerning the canonical potential, see § 3 of Chapter 1.

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \lambda_i \bar{\lambda}_j \geqq 0. \tag{2.4.1}$$

On the other hand, since $u(z)$ is harmonic in G we have $\Delta u = 0$. Recall that

$$\Delta u = \frac{1}{4} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j},$$

and so

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} = 0 \qquad \forall z \in G. \tag{2.4.2}$$

Setting $\lambda_j = 0$ in (2.4.1) for $j \neq p$ ($p = 1, \cdots, n$), we see that $\partial^2 u / \partial z_p \partial \bar{z}_p \geqq 0$ for any p. Hence by (2.4.2) it follows that, throughout the domain G,

$$\partial^2 u / \partial z_p \partial \bar{z}_p = 0 \qquad (p = 1, \cdots, n). \tag{2.4.3}$$

Now let p and q be any two of the numbers $1, \cdots, n$, and set $\lambda_j = 0$ in (2.4.1) for $j \neq p$, $j \neq q$. We get the inequality

$$2\mathrm{Re}\left\{\frac{\partial^2 u}{\partial z_p \partial \bar{z}_q} \lambda_p \bar{\lambda}_q\right\} + |\lambda_p|^2 \frac{\partial^2 u}{\partial z_p \partial \bar{z}_p} + |\lambda_q|^2 \frac{\partial^2 u}{\partial z_q \partial \bar{z}_q} \geqq 0,$$

valid for arbitrary complex λ_p and λ_q. Setting here $\lambda_p = \lambda_q = 1$ and $\lambda_p = i$, $\lambda_q = -i$, we get

$$-\frac{\partial^2 u}{\partial z_p \partial \bar{z}_p} - \frac{\partial^2 u}{\partial z_q \partial \bar{z}_q} \leqq 2\mathrm{Re}\, \frac{\partial^2 u}{\partial z_p \partial \bar{z}_p} \leqq \frac{\partial^2 u}{\partial z_p \partial \bar{z}_p} + \frac{\partial^2 u}{\partial z_q \partial \bar{z}_q}. \tag{2.4.4}$$

Hence by (2.4.2) it follows that $\mathrm{Re}\,(\partial^2 u / \partial z_p \partial \bar{z}_q) = 0$. Similarly one proves that

$$-\frac{\partial^2 u}{\partial z_p \partial \bar{z}_p} - \frac{\partial^2 u}{\partial z_q \partial \bar{z}_q} \leqq 2\,\mathrm{Im}\, \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q} \leqq \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q} + \frac{\partial^2 u}{\partial z_q \partial \bar{z}_q}. \tag{2.4.5}$$

and so $\mathrm{Im}\,(\partial^2 u / \partial z_p \partial \bar{z}_q) = 0$. We have thus proved that for arbitrary $p, q = 1, \cdots, n$ we have $\partial^2 u / \partial z_p \partial \bar{z}_q = 0$ in G. Consequently $u(z)$ is pluriharmonic. Q.E.D.

LEMMA 2.4.2. *Let the function $\Phi(z)$ be defined by $\Phi(z) = u(z) - J_\mu(z)$, where $u(z)$, μ and $J_\mu(z)$ are as in the statement of Theorem* 2.4.1. *Let*

$$\Phi_{i,j} = \partial^2 \Phi / \partial z_i \partial \bar{z}_j.$$

Then, for any i and j,

$$M_{\Phi_{i,j}}(r) \leqq C_1 + \frac{C_2}{r^2} M_{J_\mu}(2r) + C_3 r^{-2n} \mu(2r)^{18)} \tag{2.4.6}$$

for suitable constants C_1, C_2 and C_3.

---

18) The notation is as before (§ 3), i.e. $M_f(r) = \sup_{|z|<r} f^+(z)$.

PROOF. We first observe that, as follows from § 3 of Chapter 1, the function $\Phi(z)$ is harmonic. Hence $\Phi_{i,j}(z)$ is also harmonic. Consider the convolution $\Phi * \alpha_\rho$, where $\alpha_\rho(z) = \rho^{-2n}\alpha(|z|/\rho)$, $\alpha(t)$ is some infinitely differentiable function in R^1 such that $\alpha(t) = 0$ for $t \notin (0,1)$, $\alpha(t) \geqq 0$ for $t \in (0,1)$, and $\int_{C^n} \alpha(|z|)\, d\omega_z = 1$. It was shown in § 2 of Chapter 1 that the convolution of a harmonic function with the function $\alpha_\rho(z)$ is identical to the original harmonic function, so that $\Phi * \alpha_\rho \equiv \Phi$, and hence $\Phi_{i,j}(z)$ admits the representation

$$\Phi_{i,j}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(u * \alpha_\rho) - \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(J_\mu * \alpha_\rho). \tag{2.4.7}$$

Let us estimate the second term on the right of this equality. We have

$$\begin{aligned}
\left|\frac{\partial^2}{\partial z_i \partial \bar{z}_j}(J_\mu * \alpha_\rho)\right| &= \left|J_\mu * \frac{\partial^2 \alpha_\rho}{\partial z_i \partial \bar{z}_j}\right| \leqq \rho^{-2n} \max_{|z| \leqq \rho} \left|\frac{\partial^2 \alpha(|z|/\rho)}{\partial z_i \partial \bar{z}_j}\right| \cdot \int_{|\zeta| \leqq \rho} |J_\mu(z+\zeta)|\, d\omega_\zeta \\
&\leqq \text{const} \cdot \rho^{-2n-2} \int_{|\zeta| \leqq |z|+\rho} |J_\mu(\zeta)|\, d\omega_\zeta \\
&= \text{const} \cdot \rho^{-2n-2} \int_0^{\rho+|z|} \mathfrak{M}_{|J_\mu|}(0,t) t^{2n-1}\, dt \qquad (2.4.8) \\
&= \text{const} \cdot \rho^{-2n-2} \left\{ \int_0^{\rho+|z|} \mathfrak{M}_{J_\mu^+}(0,t) t^{2n-1}\, dt \right. \\
&\qquad \left. + \int_0^{\rho+|z|} \mathfrak{M}_{J_\mu^-}(0,t)\, t^{2n-1}\, dt \right\},
\end{aligned}$$

where, as usual, $J_\mu^+ = \max\{0, J_\mu\}$, $J_\mu^- = \max\{0, -J_\mu\}$, and $\mathfrak{M}_f(0,t)$ is the average of the function $f(z)$ over the sphere $|z| = t$. Now note that, as follows from Theorem 1.2.2, the functions $\mathfrak{M}_{J_\mu^+}$ and $\mathfrak{M}_{J_\mu^-}$ satisfy the relation

$$\mathfrak{M}_{J_\mu^+}(0,t) = \mathfrak{M}_{J_\mu^-}(0,t) + c \int_0^t \frac{\mu(s)}{s^{2n-1}}\, ds + J_\mu(0),$$

where $c = 2n - 2$ for $n > 1$ and $c = 1$ for $n = 1$. Therefore

$$\mathfrak{M}_{J_\mu^-}(0,t) \leqq \mathfrak{M}_{J_\mu^+}(0,t) - J_\mu(0).$$

Hence it follows by (2.4.8) that

$$\max_{|z|<r} \left|\frac{\partial^2}{\partial z_i \partial \bar{z}_j}(J_\mu * \alpha_\rho)\right| \leqq \text{const} \cdot \rho^{-2n-2} \left\{ \int_0^{r+\rho} \mathfrak{M}_{J_\mu^+}(0,t)\, t^{2n-1}\, dt - \frac{(r+\rho)^{2n}}{4n} J_\mu(0) \right\}.$$

Now, using the obvious inequality

$$\mathfrak{M}_{J_\mu^+}(0,t) \leqq M_{J_\mu^+}(t),$$

we conclude that

$$\max_{|z|\leqq r}\left|\frac{\partial^2}{\partial z_i\partial\bar z_j}(J_\mu * \alpha_\rho)\right| \leqq C_1\frac{(r+\rho)^{2n}}{\rho^{2n+2}}M_{J_\mu^+}(r+\rho) + C^2\frac{(r+\rho)^{2n}}{\rho^{2n+2}}, \qquad (2.4.9)$$

where the constants C_1 and C_2 depend only on the dimension of the space and the choice of the function α.

We now estimate the first term on the right of (2.4.7):

$$\frac{\partial^2}{\partial z_i\partial\bar z_j}(u * \alpha_\rho) = \alpha_\rho * \frac{\partial^2 u}{\partial z_i\partial\bar z_j} = \int_{C^n}\alpha_\rho(z-\zeta)\frac{\partial^2 u(\zeta)}{\partial\zeta_i\partial\bar\zeta_j}\,d\omega_\zeta.$$

The derivatives are of course to be understood in the distribution-theoretic sense. Using the relations (2.4.4) and (2.4.5), which are obviously valid for arbitrary plurisubharmonic functions, we get

$$\begin{aligned}\left|\frac{\partial^2}{\partial z_i\partial z_j}(u * \alpha_\rho)\right| &\leqq \frac{1}{4}\int_{C^n}\alpha_\rho(z-\zeta)\Delta u(\zeta)d\omega_\zeta \\ &= \frac{\theta_{2n}}{4}\int_{C^n}\alpha_\rho(z-\zeta)\,d\mu_\zeta \leqq \frac{\theta_{2n}}{4}\max_{C^n}\alpha_\rho(z)\int_{|\zeta-z|\leqq\rho}d\mu_\zeta \\ &\leqq \frac{\theta_{2n}}{4\rho^{2n}}\cdot\max_{0\leqq t\leqq 1}\alpha(t)\int_{|\zeta|\leqq|z|+\rho}d\mu_\zeta \leqq C_3\rho^{-2n}\mu(|z|+\rho),\end{aligned}$$

where θ_{2n} is the constant figuring in the definition of the associated measure (see § 2 of Chapter 1), and

$$C_3 = \frac{n\theta_{2n}}{2}\cdot\max_{0\leqq t\leqq 1}\alpha(t).$$

Hence, by (2.4.7) and (2.4.9), we see that for any i, j and any ρ,

$$M_{\Phi_{i,j}}(r) \leqq C_1\frac{(r+\rho)^{2n}}{\rho^{2n+2}} + C_2\rho^{-2n-2}(r+\rho)^{2n}M_{J_\mu^+}(r+\rho) + C_3\rho^{-2n}\mu(r+\rho).$$

Setting $r = \rho$, we get the estimate (2.4.6). Q.E.D.

We can now prove Theorem 2.4.1. Let $\Phi(z) = u(z) - J_\mu(z)$, where $J_\mu(z)$ is the canonical potential. As we mentioned in the proof of Lemma 2.4.2, the functions $\Phi_{i,j} = \partial^2\Phi/\partial z_i\partial\bar z_j$ are harmonic in C^n. Using the estimate of these functions given in Lemma 2.4.2 and the estimate (1.3.23) for the canonical potential, we conclude that the functions $\Phi_{i,j}(z)$ are of at most minimal type for order $q-1$, where q is the number figuring in the definition of the canonical potential, i.e. the smallest integer for which the integral

$$\int^\infty\frac{\mu(t)}{t^{q+2n}}\,dt$$

is convergent.[19] Hence by Lemma 1.3.1 it follows that $\Phi_{i,j}$ is a polynomial of

[19] Recall that, as was shown in § 4 of the Introduction, convergence of this integral implies that the function $\mu(t)$ is of minimal type for order $q + 2n - 1$.

degree at most $q-2$ in the variables $z_1, \cdots, z_n$ and $\bar{z}_1, \cdots, \bar{z}_n$. Moreover, it follows from the construction of the canonical kernel that the Taylor expansion of the canonical potential in the neighborhood of the origin contains no powers smaller than $q+1$, and hence all derivatives $D^{k+m}J_\mu$ of the function $J_\mu(z)$, of order $k = (k_1, \cdots, k_n)$ in the variables $z_1, \cdots, z_n$, and order $m = (m_1, \cdots, m_n)$ in the variables $\bar{z}_1, \cdots, \bar{z}_n$, with $\|k\| + \|m\| \leqq q$, satisfy the condition

$$D^{k+m}J_\mu|_{z=0} = 0. \tag{2.4.10}$$

Moreover, since by assumption the ball $|z| < t_1$ contains no Riesz masses, $J_\mu(z)$ is harmonic in this ball. Consequently $u(z)$ is also harmonic in this ball. But then it follows immediately from Lemma 2.4.1 that $u(z)$ is also pluriharmonic for $|z| < t_1$. Therefore, for $|z| < t_1$,

$$\partial^2 u/\partial z_i \partial \bar{z}_j = 0, \qquad i, j = 1, \cdots, n.$$

Hence by (2.4.10) it follows that for $\|k\| + \|m\| \leqq q-2$ and arbitrary i, j we have

$$D^{k+m}\Phi_{i,j}|_{z=0} = D^{k+m}\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} - \frac{\partial^2 J_\mu}{\partial z_i \partial z_j}\right)\Big|_{z=0} = 0.$$

As was shown above, the function $\Phi_{i,j}(z)$ is a polynomial of degree at most $q-2$; therefore $\Phi_{i,k}(z) \equiv 0$. Thus $\Phi(z)$ is pluriharmonic. This completes the proof of Theorem 2.4.1.

COROLLARY. *Let $u(z)$ be a function plurisubharmonic in C^n whose associated Riesz measure μ has finite convergence exponent and is such that $\mu(t_1) = 0$ for some $t_1 > 0$. Then the canonical potential is also a plurisubharmonic function.*

Indeed, by Theorem 2.4.1 we have $J_\mu(z) = u(z) - \Phi(z)$, and the plurisubharmonicity of $J_\mu(z)$ follows obviously from that of $u(z)$ and the pluriharmonicity of $\Phi(z)$.

To end this section, we mention that Lelong [4], [7] uses Theorem 2.4.1 to construct entire functions of minimal growth for given zero surfaces. We shall not discuss these results in detail, since they will be considered later in a special section which presents both Lelong's and other methods.

§ 5. Hartogs functions ($n > 1$)

The wide application of plurisubharmonic functions is due to their close connection with analytic functions. In § 6 of Chapter 1 we established this connection for the space C^1. We shall now study the analogous problem for functions of arbitrarily many variables. We first recall that a real-valued function $\Phi(z)$ defined in a domain $G \subset C^n$ is called a *Hartogs function* if it is in the intersection $\bigcap_{G' \Subset G} F_{G'}$, where $F_{G'}$ denotes the smallest class of functions containing all functions $\ln |f(z)|$ (where $f(z)$ is analytic in G') and closed under the following operations:

(1) formation of linear combinations:

$$\Phi(z) = \lambda_1 \Phi_1(z) + \lambda_2 \Phi_2(z) \qquad (\lambda_1, \lambda_2 \geqq 0);$$

(2) formation of least upper bounds:

$$\Phi(z) = \sup_{\alpha} \Phi_\alpha(z),$$

where $\{\Phi_\alpha(z)\}$ is a family of functions uniformly bounded above on every compact subset $K \subset G'$;

(3) formation of limits:

$$\Phi(z) = \lim_{j\to\infty} \Phi_j(z),$$

where the functions $\Phi_j(z)$ form a monotone decreasing sequence.

It follows from the properties of plurisubharmonic functions considered in §§1 and 3 that a Hartogs function which is upper semicontinuous is also plurisubharmonic. It was shown in § 5 of Chapter 1 that the converse is also valid for functions of one complex variable. The same is true of functions of several variables, as we shall presently show, though certain restrictions must be imposed on the domain in which the functions are considered. The theorem is as follows.

THEOREM 2.5.1 (BREMERMANN [2]). *Let $u(z)$ be a function plurisubharmonic in a pseudoconvex*[20] *domain $G \subset C^n$. Then $u(z)$ is a Hartogs function in G.*

The proof of this theorem for the general case is less elementary than for $n = 1$. It is based on the previously mentioned theorem of Oka (§ 1) according to which every pseudoconvex domain is a domain of holomorphy.

We first show that the domain

$$G^* = \{(z, w): z \in G, u(z) < -\ln|w|, w \in C^1\}$$

is pseudoconvex. Indeed, as is readily seen, it admits the representation $G^* = \bigcup_{N=1}^{\infty} G_N$, where

$$G_N = \{(z, w): \max[-\ln \Delta_G(z) - N, u(z) + \ln|w|] < 0\},$$

with

$$\Delta_G(z) = \min_{z' \in \partial G} |z - z'|.$$

By assumption, the functions $-\ln \Delta_G(z)$ and $u(z)$ are plurisubharmonic in G, hence also in G^*. Thus the function

$$\max[-\ln \Delta_G(z) - N, u(z) + \ln|w|]$$

is also plurisubharmonic in G^*. Consequently the domain G_N is pseudoconvex. By the Behnke-Stein Theorem,[21] the union of a monotone increasing sequence of pseudoconvex domains is again a pseudoconvex domain. The domain G^* is therefore pseudoconvex. Let $f(z, w)$ be a function holomorphic in G^* and not continuable to any larger domain. The existence of such a function follows from Oka's Theorem. By construction, G^* is a Hartogs domain, i.e. with each point (z', w') it also contains the points $\{(z, w): z = z', |w| \leqq |w'|\}$. It is known (see, for example, Šabat [1]) that functions analytic in such domains may be expanded in a Hartogs series

$$\sum_{m=0}^{\infty} a_m(z)\, w^m,$$

where $a_m(z)$ are analytic functions of z. Thus the function $f(z, w)$ admits an expansion

$$f(z, w) = \sum_{m=0}^{\infty} f_m(z) w^m$$

[20] For the definition of pseudoconvex domains, see § 1.7.

[21] See, for example, Fuks [2].

in the domain G^*, where the functions $f_m(z)$ are analytic in G. Moreover, the radius $R_f(z')$ of the greatest disk $\{(z, w): z = z', |w| < R\}$ in which the function $f(z, w)$ is holomorphic is defined by

$$-\ln R_f(z') = \overline{\lim_{z \to z'}}\ \overline{\lim_{m \to \infty}}\ \frac{1}{m} \ln |f_m(z)|.$$

Since $f(z, w)$ has no analytic continuation outside G^*, it follows that $-\ln R_f(z) = u(z)$, and so

$$u(z) = \overline{\lim_{z \to z'}}\ \overline{\lim_{m \to \infty}}\ \frac{1}{m} \ln |f_m(z)|,$$

or, what is the same,

$$u(z) = \lim_{\varepsilon \to 0} \sup_{|z'| < \varepsilon} \lim_{j \to \infty} \sup_{m \geq j} \frac{1}{m} \ln |f_m(z' + z)|.$$

Consequently $u(z)$ is a Hartogs function. Q.E.D.

Note that the pseudoconvexity condition figuring in this theorem is crucial. This is indicated by the following example, due to Bremermann [3].

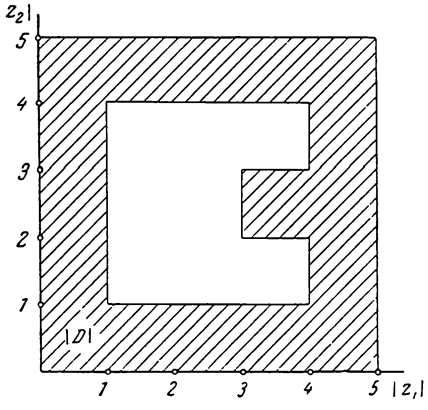

FIGURE 1

Let D be the bicircular domain whose image in the absolute quadrant is illustrated in Figure 1. Set $u(z) = 0$ for $z \in D \backslash E$, where $E = \{(z_1, z_2): 3 < |z_1| < 4, 2 < |z_2| < 3\}$, and $u(z) = (16 - |z_1|^2)^4$ for $z \in E$. Then for all $\alpha_1, \ldots, \alpha_n \in C_1$ we have

$$\sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial z_i\, \partial \bar{z}_j} \alpha_i \bar{\alpha}_j = 0 \qquad \forall z \in D \backslash E,$$

$$\sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial z_i\, \partial \bar{z}_j} \alpha_i \bar{\alpha}_j = 16(16 - |z_1|^2)(|z_1|^2 - 4)|\alpha_1|^2 > 0 \qquad \forall z \in E.$$

Thus $u(z)$ is plurisubharmonic. On the other hand, it cannot be a Hartogs function in D. Indeed, every function analytic in D has an analytic continuation to the polydisk

$$\mathscr{E} = \{(z_1, z_2): |z_1| < 5, |z_2| < 5\}.$$

Thus any Hartogs function in D is continuable as a Hartogs function to this polydisk. But this is impossible for the function $u(z)$, since $u(z) > 0$ for $z \in E$, $u(z) = 0$ in the neighborhood of the boundary of $\mathscr{E}$, and thus any continuation of $u(z)$ as a Hartogs function to $\mathscr{E}$ would contradict the maximum principle, whose validity for Hartogs functions is obvious. Thus the function $u(z)$, though plurisubharmonic, is not a Hartogs function.

§ 6. Functions of class $\mathfrak{A}$

Let $\mathfrak{A}$ denote the class of functions $\Phi(r) = \Phi(r_1, \cdots, r_n)$, defined for $r_1 \geqq 0, \cdots, r_n \geqq 0$, such that the function $\Phi(|z_1|, \cdots, |z_n|)$ is plurisubharmonic in C^n.[22] As was shown in § 1, this class includes such familiar functions as $\ln^+ M_f(r)$ and $\mathfrak{M}_u^c(z, r)$, where

$$M_f(r) = \max_{|z_i| = r_i, i = 1, \cdots, n} |f(z)|,$$

the function $f(z)$ is entire, i.e. analytic in C^n, and $u(z)$ is a plurisubharmonic function in C^n. We wish to investigate the growth of functions $\Phi(r) \in \mathfrak{A}$. In so doing,

[22] This class of functions was first introduced in Ronkin [9].

we shall make essential use of the fact that, by Theorem 2.1.2, the functions $\Phi(r)$ are, first, monotone nondecreasing in each variable, and, second, convex in $\ln r_1, \cdots, \ln r_n$, so that for any $t \in R_+^n$, any $s \in R_+^n$ and any $\lambda \in [0, 1]$

$$\Phi(t_1^{\lambda} s_1^{\mu}, \cdots, t_n^{\lambda} s_n^{\mu}) \leqq \lambda \Phi(t_1, \cdots, t_n) + \mu \Phi(s_1, \cdots, s_n), \tag{2.6.1}$$

where $\mu = 1 - \lambda$.

1. *Order and type.* Let G be a complete domain in R_+^n. Let RG denote the domain obtained from G by a similarity transformation about the origin, with ratio of similitude R. We also denote

$$M_{\Phi}(R; G) = \max_{r \in RG} \Phi(r) \qquad (\Phi(r) \in \mathfrak{A}).$$

We define the order $\rho(\Phi)$ of a function $\Phi \in \mathfrak{A}$, $\Phi(r) \not\equiv \text{const}$, to be the order of the function $M_{\Phi}(R; G)$:

$$\rho(\Phi) = \varlimsup_{R \to \infty} \frac{\ln M_{\Phi}(R; G)}{\ln R}.$$

This definition is legitimate, since, as we shall immediately show, the order of $M_{\Phi}(R; G)$ is independent of the choice of the domain G. Indeed, for any pair of complete domains G_1 and G_2 there clearly exist numbers $\alpha > 0$ and $\beta > 0$ such that $\alpha G_1 \subset G_2 \subset \beta G_1$, and thus

$$M_{\Phi}(R; \alpha G_1) \leqq M_{\Phi}(R; G_2) \leqq M_{\Phi}(R; \beta G_1). \tag{2.6.2}$$

Now, for any $\lambda > 0$,

$$M_{\Phi}(R; \lambda G) = M_{\Phi}(\lambda R; G).$$

Therefore

$$\varlimsup_{R \to \infty} \frac{\ln M_{\Phi}(R; \lambda G)}{\ln R} = \varlimsup_{R \to \infty} \frac{\ln M_{\Phi}(R; G_2)}{\ln R}.$$

Hence by (2.6.2) it follows that

$$\varlimsup_{R \to \infty} \frac{\ln M_{\Phi}(R; G_1)}{\ln R} = \varlimsup_{R \to \infty} \frac{\ln M_{\Phi}(R; G_2)}{\ln R}$$

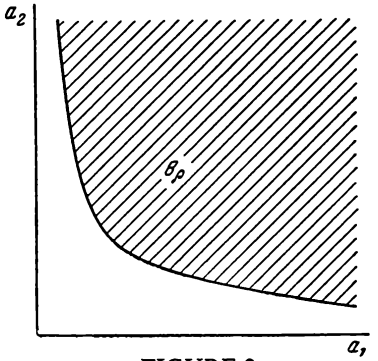

FIGURE 2

and hence the order of the function $M_{\Phi}(R; G)$ is the same for any complete domain G. As simple examples show, the analogous statement is not valid for the type of the function $M_{\Phi}(R; G)$. We therefore adopt the following definition:

The *G-type* $\sigma_G(\Phi)$[23] of a function $\Phi \in \mathfrak{A}$ for order ρ is defined as the type of the function $M_{\Phi}(R; G)$ for order ρ:

[23] This definition of order and G-type is due to Gol'dberg, who considered the functions $\Phi(r) = \ln M_f(r_1, \cdots, r_n)$ (where $f(z)$ is an entire function). See Gol'dberg [1].

$$\sigma_G(\Phi) = \overline{\lim_{R\to\infty}} \frac{M_\Phi(R; G)}{R^\rho}.$$

Just as in the one-variable case, the order and G-type are the simplest and nevertheless most penetrating characteristics of the growth of functions. More refined characterizations can be obtained by direct comparison of the growth of the function $\Phi(r)$ with that of the functions $b_1 r_1^{a_1} + \cdots + b_n r_n^{a_n}$.

2. *Associated orders.* Let $\Phi(r) \in \mathfrak{A}$ be a function of finite order $\rho(\Phi)$. Let $B_\rho = B_\rho(\Phi)$ (see Figure 2) denote the set of all points $a \in R_+^n$ such that, for $|r| \to \infty$,

$$\Phi(r) < r_1^{a_1} + \cdots + r_n^{a_n}. \tag{2.6.3}$$

It is obvious that for $a' \in B_\rho$ the set B_ρ contains the entire hyperoctant $\{a: a_i > a_i',\ i = 1,2, \cdots, n\}$. Henceforth sets in R^n possessing this property will be called *octant-like*. On the other hand, if $a' \notin B_\rho$, then any point $a \in R_+^n$ with $a_i \leqq a_i'$ is also not in B_ρ. Hence the boundary points of the set $B_\rho(\Phi)$ form a certain hypersurface $S_\rho = S_\rho(\Phi)$. This hypersurface (see Figure 2) generally divides the hyperoctant R_+^n into two parts,[24] in one of which inequality (2.6.3) holds, while in the other it is false. Thus the hypersurface S_ρ characterizes the growth of the function $\Phi(r)$. We shall call it the *hypersurface of associated orders*[25] *of* $\Phi(r)$, and any system of numbers $\rho_1, \cdots, \rho_n$ such that $(\rho_1, \cdots, \rho_n) \in S_\rho$ will be called a *system of associated orders* of the function. If the numbers $\rho_1, \cdots, \rho_n$ are known to be positive, then, as is readily seen, *they form a system of associated orders if and only if*

$$\overline{\lim_{|r|\to\infty}} \frac{\ln \Phi^+(r)}{\ln (r_1^{\rho_1} + \cdots + r_n^{\rho_n})} = 1.^{26}$$

EXAMPLE 1. Let $\Phi(r) = r_1^{\alpha_1} + \cdots + r_n^{\alpha_n}$, where $\alpha_1 > 0, \cdots, \alpha_n > 0$. It is clear that

$$\overline{\lim_{|r|\to\infty}} \frac{\ln (r_1^{\alpha_1} + \cdots + r_n^{\alpha_n})}{\ln (r_1^{\rho_1} + \cdots + r_n^{\rho_n})} = 1$$

if and only if $\rho_i \geqq \alpha_i$ for any i and the equality $\rho_i = \alpha_i$ holds for at least one value of i. In this case, therefore, the hypersurface of associated orders is the boundary of the hyperoctant $\{a: a_1 > \alpha_1, \cdots, a_n > \alpha_n\}$.

EXAMPLE 2. Let $\Phi(r) = r_1 \cdots r_n$. Since, for any nonnegative $a_1, \cdots, a_n$,

$$r_1 r_2 \cdots r_n = \prod_{i=1}^{n} (r_i^{a_i})^{1/a_i} \leqq (r_1^{a_1} + \cdots + r_n^{a_n}) \textstyle\sum_1^n a_i^{-1},$$

it follows that all points $(a_1, \cdots, a_n) \in R_+^n$ satisfying the condition $\sum_1^n a_i^{-1} \leqq 1$ are

[24] One of these parts may be empty.

[25] For functions $\Phi(r) = \ln M_f(r_1, \cdots, r_n)$, where $f(z)$ is an entire function, hypersurfaces of associated orders were introduced by the author [4]; see also Fuks [1].

[26] In the above-mentioned case $\Phi(r) = \ln M_f(r)$, Gol'dberg introduced systems of positive associated orders defined by this equality (see Gol'dberg [1]).

in the set B_ρ. On the other hand, if $\sum_1^n a_i^{-1} > 1$, then $(a_1, \cdots, a_n) \notin B$. Indeed, in this case, setting $r_i = t^{1/a_i}$, $i = 1, \cdots, n$, we see that for sufficiently large values of t

$$r_1 r_2 \cdots r_n = t \sum_1^n \frac{1}{a_i} > nt = r_1^{a_1} + \cdots + r_n^{a_n},$$

i.e. inequality (2.6.3) is false, and thus $(a_1, \cdots, a_n) \notin B$. Consequently the hypersurface of associated orders of this function is

$$\left\{a: a \in R_+^n, \sum_{i=1}^n \frac{1}{a_i} = 1\right\}.$$

Using the properties of functions of class $\mathfrak{A}$, it is not hard to establish a geometric description of the hypersurface of associated orders. This is given by the following theorem.

THEOREM 2.6.1 (RONKIN [9]). *Let $\Phi(r) \in \mathfrak{A}$, and let $\mathring{B}_\rho$ be the domain consisting of the interior points of the set $B_\rho(\Phi)$. Then the domain B_ρ^{-1}, i.e. the image of $\mathring{B}_\rho$ under the transformation $a_i' = 1/a_i$, is a complete convex domain.*

PROOF. That the domain B_ρ^{-1} is complete is obvious. It remains to prove that it is convex. To do this, we take two arbitrary points a and b in $\mathring{B}_\rho$ and estimate the function $\Phi(r)$ by (2.6.1), setting

$$t_i = r_i^{a_i/(\lambda a_i + \mu b_i)}, \qquad s_i = r_i^{b_i/(\lambda a_i + \mu b_i)}, \qquad i = 1, \cdots, n.$$

It follows, via the definition of the set B_ρ, that for $|r| \to \infty$

$$\Phi(r) \leqq \lambda \Phi(t) + \mu \Phi(s) < \lambda \sum_{i=1}^n t_i^{b_i} + \mu \sum_{i=1}^n s_i^{a_i} = \sum_{i=1}^n r_i^{1/(\lambda/b_i + \mu/a_i)}.$$

Consequently, for any λ and μ, $\lambda \geqq 0$, $\mu \geqq 0$, $\lambda + \mu = 1$, the point $(\lambda/b_1 + \mu/a_1, \cdots, \lambda/b_n + \mu/a_n)$, i.e. any point of the segment joining points $(b_1^{-1}, \cdots, b_n^{-1})$ and $(a_1^{-1}, \cdots, a_n^{-1})$ of the domain B_ρ^{-1}, is also in B_ρ^{-1}. Thus the domain B_ρ^{-1} is convex. Q.E.D.

Later on (§ 1 of Chapter 3) we shall prove that the convexity and completeness of a domain $B \subset R_+^n$ are not only necessary but also sufficient for the boundary S of the domain $B^{-1} = \{a: (a_1^{-1}, \cdots, a_n^{-1}) \in B\}$ to be the hypersurface of associated orders for some function $\Phi(r)$ of class $\mathfrak{A}$, indeed of a certain subclass of $\mathfrak{A}$.

The hypersurface of associated orders uniquely determines the order $\rho(\Phi)$ of the function. Indeed, let $(t, t, \cdots, t) \in S_\rho(\Phi)$. Then for $|r| \to \infty$ and any $\varepsilon > 0$ we have the inequality

$$\Phi(r) < r_1^{t+\varepsilon} + \cdots + r_n^{t+\varepsilon}$$

and therefore

$$\Phi(R, R, \cdots, R) < nR^{t+\varepsilon} < R^{t+2\varepsilon}. \tag{2.6.4}$$

On the other hand, the inequality $\Phi(R, \cdots, R) \leqq R^{t-\varepsilon}$ cannot hold for all sufficiently large R, since then we would have an inequality

$$\Phi(r) \leqq \Phi(|r|, \cdots, |r|) \leqq |r|^{t-\varepsilon} < r_1^{t-\varepsilon/2} + \cdots + r_n^{t-\varepsilon/2},$$

contradicting the fact that $(t, \cdots, t)$ is in the hypersurface of associated orders. Consequently the order of the function $\Phi(R, \cdots, R)$ is t. But $\Phi(R, \cdots, R) = M_\Phi(R;G)$, where $G = \{r: 0 \leqq r_i \leqq 1, i = 1, \cdots, n\}$, and thus $t = \rho(\Phi)$, i.e. the order of the function $\Phi(r)$ equals the value of the parameter t defining the intersection of the hypersurface $S_\rho(\Phi)$ with the ray $\{a: a \in R_+^n, a_i = t, i = 1, \cdots, n, 0 \leqq t < \infty\}$.

3. *Associated types.* Let $\rho_1 > 0, \cdots, \rho_n > 0$ be a system of associated orders of a function $\Phi(r) \in \mathfrak{A}$. Let us compare the growth of the function $\Phi(r)$ with that of functions of the form $b_1 r_1^{\rho_1} + \cdots + b_n r_n^{\rho_n}$. The procedure will be analogous to that whereby we compared the growth of $\Phi(r)$ with that of $r_1^{a_1} + \cdots + r_n^{a_n}$ in the preceding subsection.

Let $B_\sigma = B_\sigma(\Phi, \rho)$ denote the set of all points $b \in R_+^n$ such that $\Phi(r) < b_1 r_1^{\rho_1} + \cdots + b_n r_n^{\rho_n}$, for $|r| \to \infty$. Like the set B_ρ, the set B_σ is octant-like and its boundary $S_\sigma = S_\sigma(\Phi, \rho)$ is a hypersurface. It is clear from the construction of S_σ that it characterizes the growth of the function $\Phi(r)$. Let us call S_σ the *hypersurface of associated types for associated orders* $\rho_1, \cdots, \rho_n$; any system of numbers $\sigma_1, \cdots, \sigma_n$ such that $(\sigma_1, \cdots, \sigma_n) \in S$ will be called a *system of associated types for associated orders* $\rho_1, \cdots, \rho_n$. If the numbers $\sigma_1, \cdots, \sigma_n$ are positive, then, as is easily seen, *they form a system of associated types for associated orders* $\rho_1, \cdots, \rho_n$ *if and only if*

$$\overline{\lim_{|r| \to \infty}} \frac{\Phi(r)}{\sigma_1 r_1^{\rho_1} + \cdots + \sigma_n r_n^{\rho_n}} = 1.$$

In the sequel we shall often omit the phrase "for associated orders $\rho_1, \cdots, \rho_n$" in order to simplify the notation.

Like the hypersurface S_ρ, the hypersurface S_σ possesses very simple geometric properties.

THEOREM 2.6.2 (RONKIN [9],[1]). *Let $\Phi(r) \in \mathfrak{A}$, and let $\mathring{B}_\sigma$ be the domain consisting of the interior points of the corresponding set B_σ. Then the domain* $\ln B_\sigma$, *i.e. the image of $\mathring{B}_\sigma$ under the transformation $b_i' = \ln b_i$, $i = 1, \cdots, n$, is octant-like and convex.*

PROOF. That $\ln B_\sigma$ is octant-like is obvious. To prove convexity, we consider arbitrary points $b \in \mathring{B}_\rho$ and $b' \in \mathring{B}_\rho$ and estimate the function $\Phi(r)$ by (2.6.1), setting

$$t_i = r_i \left(\frac{b_i'}{b_i}\right)^{\mu/\rho_i}, \quad s_i = r_i \left(\frac{b_i}{b_i'}\right)^{\lambda/\rho_i}, \qquad i = 1, \cdots, n.$$

We then conclude, via the definition of the set B_σ, that

$$\Phi(r) \leqq \lambda\Phi(t) + \mu\Phi(s) < \lambda \sum_{i=1}^{n} b_i t_i^{\rho_i} + \mu \sum_{i=1}^{n} b_i' s_i^{\rho_i} = \sum_{i=1}^{n} b_i'^{\mu} b_i^{\lambda} r_i^{\rho_i}.$$

Thus $(b_1'^{\mu}b_1^{\lambda}, \cdots, b_n'^{\mu}b_n^{\lambda}) \in B_\sigma$, and the points $(\mu \ln b_1' + \lambda \ln b_1, \cdots, \mu \ln b_n' + \lambda \ln b_n)$, i.e. all points on the segment joining the points $(\ln b_1', \cdots, \ln b_n') \in \ln B_\sigma$ and $(\ln b_1, \cdots, \ln b_n) \in \ln B_\sigma$, lie in the domain $\ln B_\sigma$. Thus the domain $\ln B_\sigma$ is convex. Q.E.D.

Later on (§ 1, Chapter 3), we shall show that the properties of the domain $\mathring{B}_\sigma$ established in Theorem 2.6.2 are characteristic.

4. *Order of the function $\Phi(r_1, \cdots, r_n)$ in one of the variables $r_1, \cdots, r_n$.* It is clear from the definitions that the associated orders and types characterize the growth of the functions with respect to all the variables jointly. However, in many cases one needs some characterization of the growth of a function relative to one of its variables.

DEFINITION. The *order $\bar{\rho}_i = \bar{\rho}_i(\Phi)$ of a function $\Phi(r) \in \mathfrak{A}$ in the variable r_i* is its order as a function of r_i, the other variables $r_j, j \neq i$, being fixed:

$$\rho_i(\Phi) = \overline{\lim_{r_i \to \infty}} \frac{\ln \Phi(r)}{\ln r_i}.$$

We claim that this definition is legitimate: the order $\bar{\rho}_i(\Phi)$ is independent of the values assigned to the fixed variables $r_j, j \neq i$. Without loss of generality, let $i = n$. Set

$$\rho(r_1, \cdots, r_{n-1}) = \overline{\lim_{r_n \to \infty}} \frac{\ln \Phi(r)}{\ln r_n}.$$

It is obvious that $\rho(r_1, \cdots, r_{n-1})$ is monotone nondecreasing, and that for any $\varepsilon > 0$ and all $r_1 > 0, \cdots, r_n > 0$

$$\Phi(r) \leqq C_\varepsilon(r_1, \cdots, r_{n-1}) + r_n^{\rho(r_1, \cdots, r_{n-1}) + \varepsilon}$$

with some function $C_\varepsilon(r_1, \cdots, r_{n-1}) < \infty$. Let us estimate the function $\Phi(r)$ by inequality (2.6.1), with $t_i = r_i$, $s_i = (R_i/r_i^{\lambda})^{1/\mu}$, $i = 1, \cdots, n-1$, $s_n = 1$ and $t_n = R_n^{1/\lambda}$. Then

$$\Phi(R_1, \cdots, R_n) \leqq \lambda\Phi\left(r_1, \cdots, r_{n-1}, R_n^{1/\lambda}\right) + \mu\Phi\left(\frac{R_1^{1/\mu}}{r_1^{\lambda/\mu}}, \cdots, \frac{R_{n-1}^{1/\mu}}{r_{n-1}^{\lambda/\mu}}, 1\right)$$

$$\leqq C_\varepsilon(r_1, \cdots, r_{n-1}) + R_n^{(\rho(r_1, \cdots, r_{n-1}) + \varepsilon)/\lambda} + \Phi\left(R_1^{1/\lambda} r_1^{-\lambda/\mu}, \cdots, R_{n-1}^{1/\lambda} r_{n-1}^{-\lambda/\mu}, 1\right).$$

Consequently for any $\lambda \in (0, 1)$, $\varepsilon > 0$, $r_i > 0$ and $R_i > 0$

$$\rho(R_1, \cdots, R_{n-1}) \leqq \frac{1}{\lambda} \{\rho(r_1, \cdots, r_{n-1}) + \varepsilon\}.$$

Hence, since the numbers $\varepsilon > 0$ and $\lambda \in (0, 1)$ are arbitrary, we see that for any

$r_i > 0$ and $R_i > 0$, $i = 1, \cdots, n-1$,

$$\rho(R_1, \cdots, R_{n-1}) \leqq \rho(r_1, \cdots, r_{n-1}).$$

Consequently $\rho(r_1, \cdots, r_{n-1}) \equiv \text{const}$, and we can speak of the order of the function $\Phi(r)$ in the variable r_n, irrespective of the values assigned to the fixed variables $r_1, \cdots, r_{n-1}$.

Since functions of $\mathfrak{A}$ are monotone nondecreasing in each of their variables, the order $\bar{\rho}_i(\Phi)$ for any i is related to the order $\rho(\Phi)$ by the inequality

$$\bar{\rho}_i(\Phi) \leqq \rho(\Phi). \tag{2.6.5}$$

It may even happen that for some i we have $\bar{\rho}_i(\Phi) < \infty$, whereas $\rho(\Phi) = \infty$. But this situation cannot occur if $\bar{\rho}_i(\Phi) < \infty$ for all values of i. This will be clear from the following theorem.

THEOREM 2.6.3.[27] $\rho(\Phi) \leqq \sum_1^n \bar{\rho}_i(\Phi)$ *for any function* $\Phi(r) \in \mathfrak{A}$.

PROOF. Since the function $\Phi(r)$ is convex in $\ln r_1, \cdots, \ln r_n$, it follows that, besides inequality (2.6.1), the following more general inequality is obviously valid:

$$\Phi\left(t_{11}^{\lambda_1} \cdot t_{12}^{\lambda_2} \cdots t_{1n}^{\lambda_n}, \cdots, t_{n1}^{\lambda_n} \cdots t_{nn}^{\lambda_n}\right) \leqq \sum_{i=1}^{n} \lambda_i \Phi(t_{i1}, \cdots, t_{in}),$$

where $\lambda_i \geqq 0$, $\lambda_1 + \cdots + \lambda_n = 1$ and $t_{ij} > 0$. Let us estimate the function $\Phi(R, \cdots, R)$ by this inequality, setting $t_{i,j} = 1$ for $i \neq j$ and $t_{ii} = R^{1/\lambda_i}$, $i = 1, \cdots, n$. Then

$$\Phi(R, \cdots, R) \leqq \sum_{i=1}^{n} \lambda_i \Phi(1, \cdots, 1, R^{1/\lambda_i}, 1, \cdots, 1).$$

Now, by the definition of the order $\bar{\rho}_i$, for any $\varepsilon > 0$ and all $r_i > 0$ we have

$$\Phi(1, \cdots, 1, r_i, 1, \cdots, 1) \leqq C_\varepsilon^{(i)} + r_i^{\bar{\rho}_i + \varepsilon}$$

with some constant $C_\varepsilon^{(i)}$, $i = 1, \cdots, n$. Therefore for any $\varepsilon > 0$ there exists a constant C_ε such that for any $R > 0$

$$\Phi(R, \cdots, R) \leqq C_\varepsilon + \sum_{i=1}^{n} R^{(\bar{\rho}_i + \varepsilon)/\lambda_i}.$$

Setting

$$\lambda_i = \frac{\bar{\rho}_i + \varepsilon}{\bar{\rho}_1 + \cdots + \bar{\rho}_n + n\varepsilon},$$

we get the inequality

[27] This theorem was first proved, for entire functions, by E. Borel [1] and Valiron [1]. See also Maergoïz [1], [2].

$$\Phi(R, \cdots, R) \leqq C_\varepsilon + nR\sum_1^n \bar{\rho}_i + n\varepsilon$$

which immediately implies that $\rho(\Phi) \leqq \sum_1^n \bar{\rho}_i(\Phi)$. Q.E.D.

It readily follows from the definition of associated orders that, besides inequality (2.6.5), we also have the inequality $\bar{\rho}_i \leqq \rho_i$ for any i, where $\rho_1, \cdots, \rho_n$ is an arbitrary system of associated orders. Moreover, we have

THEOREM 2.6.4 (RONKIN [9]). *Let the function $\Phi(r) \in \mathfrak{A}$ have order $\rho(\Phi) = \rho < \infty$. Then*

$$\bar{\rho}_i = \inf_{(\rho_1, \cdots \rho_n) \in S_\rho(\Phi)} \rho_i.$$

PROOF. To simplify the notation, let $i = n$. By the definition of the numbers $\bar{\rho}_n$ and ρ, for any $\varepsilon > 0$ and all sufficiently large $R > 0$ and $R_n > 0$ we have

$$\Phi(R, \cdots, R) < R^{\rho+\varepsilon}, \qquad \Phi(1, \cdots, 1, R_n) < R_n^{\bar{\rho}_n+\varepsilon}.$$

Hence by inequality (2.6.1), with $R = \max[R_1, \cdots, R_{n-1}]$, $t_i = R^{1/\lambda}$, $s_i = 1$, $i = 1, \cdots, n-1$, $t_n = 1$ and $S_n = R^{1/\mu}$, and by the monotonicity of the function $\Phi(r)$, it follows that

$$\Phi(R_1, \cdots, R_n) \leqq \Phi(R, \cdots, R, R_n) \leqq \lambda\Phi(R^{1/\lambda}, \cdots, R^{1/\lambda}, 1) + \mu\Phi(1, \cdots, 1, R_n^{1/\mu})$$

$$< \lambda R^{(\rho+\varepsilon)/\lambda} + \mu R_n^{(\bar{\rho}_n+\varepsilon)/\mu} < \sum_{i=1}^{n-1} R_i^{(\rho+\varepsilon)/\lambda} + R_n^{(\bar{\rho}_n+\varepsilon)/\mu}.$$

This means that

$$\left(\frac{\rho+\varepsilon}{\lambda}, \cdots, \frac{\rho+\varepsilon}{\lambda}, \frac{\bar{\rho}_n+\varepsilon}{\mu}\right) \in B_\rho(\Phi)$$

and hence there exists a point $(\rho_1, \cdots, \rho_n) \in S_\rho(\Phi)$ such that $\rho_n \leqq (\bar{\rho}_n + \varepsilon)/\mu$. Since the numbers $\varepsilon > 0$ and $\mu \in (0, 1)$ are arbitrary, this implies that

$$\inf_{(\rho_1, \cdots, \rho_n) \in S_\rho} \rho_n \leqq \bar{\rho}_n.$$

The converse inequality is obvious. Consequently

$$\bar{\rho}_n = \inf_{(\rho_1, \cdots, \rho_n) \in S_\rho} \rho_n.$$

Q.E.D.

5. *Type of the function $\Phi(r_1, \cdots, r_n)$ in one of the variables $r_1, \cdots, r_n$.* Let $\Phi(r) \in \mathfrak{A}$ be a function of finite order $\bar{\rho}_i$ in the variable r_i. Let

$$\sigma_i(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n; \Phi, \bar{\rho}_i) = \sigma_i(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n)$$

denote the type of the function $\Phi(r_1, \cdots, r_n)$ for order $\bar{\rho}_i$ in the variable r_i for fixed $r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n$. In other words,

$$\sigma_i(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n) = \overline{\lim_{r_i \to \infty}} \frac{\Phi(r)}{r_i^{\bar{\rho}_i}}.$$

We define the *upper type* of the function $\Phi(r)$ for order $\bar{\rho}_i$ in the variable r_i to be the number $\bar{\sigma}_i = \bar{\sigma}_i(\Phi, \bar{\rho}_i)$ defined by

$$\bar{\sigma}_i = \sup_{(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n)} \sigma_i(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n).$$

In subsequent investigation of the properties of $\sigma_i(r_1, \cdots, r_{i-1}, r_{i+i}, \cdots, r_n)$ we shall set $i = n$ in order to simplify the notation.

We first note that, unlike the order $\rho(r_1, \cdots, r_{n-1})$, the type $\sigma_n(r_1, \cdots, r_{n-1})$ need not be identically equal to the upper type $\bar{\sigma}$. Thus, if $\Phi(r_1, r_2) = r_1 r_2$, we have $\sigma_2(r_1) = r_2$, but $\bar{\sigma}_2 = \infty$. Further, note that the function $\sigma_n(r_1, \cdots, r_{n-1})$ is monotone nondecreasing in each of its variables. This is an obvious consequence of the analogous property of the function $\Phi(r)$. It is also readily seen that *the type* $\sigma_n(r_1, \cdots, r_{n-1})$ *is a convex function of the variables* $\ln r_1, \cdots, \ln r_n$. Indeed, setting $t_i = r_i$ and $s_i = R_i$ for $i = 1, \cdots, n-1$, and $t_n = s_n = R$ in (2.6.1), we see that for any $\lambda \geqq 0$, $\mu \geqq 0$, $\lambda + \mu = 1$, $r_i > 0$, $R_i > 0$, $i = 1, \cdots, n-1$, and $R > 0$ we have the inequality

$$\Phi(r_1^{\lambda} R_1^{\mu}, \cdots, r_{n-1}^{\lambda} R_{n-1}^{\mu}, R) \leqq \lambda \Phi(r_1, \cdots, r_{n-1}, R) + \mu \Phi(R_1, \cdots, R_{n-1}, R).$$

Consequently

$$\sigma_n(r_1^{\lambda} R_1^{\mu}, \cdots, r_{n-1}^{\lambda} R_{n-1}^{\mu}) \leqq \lambda \sigma_n(r_1, \cdots, r_{n-1}) + \mu \sigma_n(R_1, \cdots, R_{n-1}).$$

and so $\sigma_n(r_1, \cdots, r_{n-1})$ is a convex function of $\ln r_1, \cdots, \ln r_{n-1}$.

It is readily seen that, since the function $\sigma_n(r_1, \cdots, r_{n-1})$ is convex and monotone nondecreasing, it follows that *if* $\bar{\sigma}_n < \infty$ *the function* $\sigma_n(r_1, \cdots, r_{n-1})$ *is identically equal to* $\bar{\sigma}_n$.

The relationship of the types $\bar{\sigma}_i$ to the associated types is far weaker than that of the orders $\bar{\rho}_i$ to the associated orders. For one thing, they do not satisfy analogs of Theorems 2.6.3 and 2.6.4. In addition, these theorems are not valid even if the class $\mathfrak{A}$ is replaced by a certain subclass, intimately related to entire functions in C^n. Examples will be given in Chapter 3, which is devoted to the study of entire functions.

A very simple upper bound can be given for the type $\sigma_n(r_1, \cdots, r_{n-1})$ of functions $\Phi(r)$ of finite order. We have

THEOREM 2.6.5 (RONKIN [9]). *Let* $\Phi(r) \in \mathfrak{A}$ *be a function of order* $\rho(\Phi) = \rho < \infty$, *and order* $\bar{\rho}_n(\Phi) = \bar{\rho}_n > 0$ *in the variable* r_n. *If there exist* $r_1^0 > 0, \cdots, r_{n-1}^0 > 0$ *such that* $\sigma_n(r_1^0, \cdots, r_{n-1}^0) = \gamma < \infty$, *then the following estimate is valid for any* $R > r_0$, *where* $r_0 = \min r_i^0$:

$$\sigma_n(R, \cdots, R) \leqq \gamma (R / r_0)^{\rho - \bar{\rho}_n}.$$

PROOF. As before, we estimate the function $\Phi(r)$ by inequality (2.6.1). To this end, take

$$t_i = R^{\alpha_i/(\lambda\alpha_i+\mu\beta_i)}, \qquad s_i = R^{\beta_i/(\lambda\alpha_i+\mu\beta_i)}, \qquad i = 1, \cdots, n-1,$$
$$t_n = r_n^{\alpha_n/(\lambda\alpha_n+\mu\beta_n)}, \qquad s_n = r_n^{\beta_n/(\lambda\alpha_n+\mu\beta_n)},$$

where

$$\alpha_n = \bar{\rho}_n - \varepsilon_1, \qquad 0 < \varepsilon_1 < \bar{\rho}_n, \qquad \beta_n = \rho,$$
$$\alpha_i/\beta_i = \ln r_n - C(R), \qquad i = 1, \cdots, n-1,$$
$$C(R) = -1 + \frac{\rho}{\bar{\rho}_n - \varepsilon_1} \ln R,$$
$$\lambda = \frac{C(R)}{\ln r_n}, \qquad \mu = 1 - \lambda = \frac{\ln r_n - C(R)}{\ln r_n}.$$

We may assume here, without loss of generality, that $r_n \geqq R^{\rho/(\bar{\rho}_n-\varepsilon_1)}/e$ and $R \geqq e^{\bar{\rho}_n/\rho}$. It then follows from (2.6.1) that

$$\Phi(R, \cdots, R, r_n) \leqq \lambda\Phi\left(R^{\frac{\alpha_1}{\lambda\alpha_1+\mu\beta_1}}, \cdots, R^{\frac{\alpha_{n-1}}{\lambda\alpha_{n-1}+\mu\beta_{n-1}}}, r_n^{\frac{\alpha_n}{\lambda\alpha_n+\mu\beta_n}}\right)$$
$$+ \mu\Phi\left(R^{\frac{\beta_1}{\lambda\alpha_1+\mu\beta_1}}, \cdots, R^{\frac{\beta_{n-1}}{\lambda\alpha_{n-1}+\mu\beta_{n-1}}}, r^{\frac{\beta_n}{\lambda\alpha_n+\mu\beta_n}}\right) = \mathrm{I} + \mathrm{II}.$$

Our subsequent estimates are based on the fact that, by definition of the numbers ρ, $\bar{\rho}_n$ and γ, for any $\varepsilon > 0$ there exist constants $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(2)}$ such that, for all $r_1 \geqq 0, \cdots, r_n \geqq 0$,

$$\Phi(r) \leqq C_\varepsilon^{(1)} + r_1^{\rho+\varepsilon} + \cdots + r_n^{\rho+\varepsilon} \tag{2.6.6}$$

and, for any $r_n \geqq 0$,

$$\Phi(r_1^0, \cdots, r_{n-1}^0, r_n) \leqq C_\varepsilon^{(2)} + (\gamma + \varepsilon) r_n^{\bar{\rho}_n}. \tag{2.6.7}$$

It follows from (2.6.6) that

$$\mathrm{I} \leqq \left\{\lambda C_\varepsilon^{(1)} + \lambda \sum_{i=1}^{n-1} R^{\frac{\alpha_i(\rho+\varepsilon)}{\lambda\alpha_i+\mu\beta_i}}\right\} + \lambda r^{\frac{\alpha_n(\rho+\varepsilon)}{\lambda\alpha_n+\mu\beta_n}} = \mathrm{I}_1 + \mathrm{I}_2.$$

Now, in view of the special choice of the numbers λ, μ, α_i, β_i, we get

$$\mathrm{I}_1 \leqq C_\varepsilon^{(1)} + (n-1)\exp\left\{\frac{(\rho+\varepsilon)\ln R}{\frac{C(R)}{\ln r_n} + \frac{\ln r_n - C(R)}{\ln r_n} \cdot \frac{1}{\ln r_n - C(R)}}\right\}$$
$$= C_\varepsilon^{(1)} + (n-1) R^{(\rho+\varepsilon)\ln r_n/C(R)+1} = C_\varepsilon^{(1)} + (n-1) R^{(\rho_n+\varepsilon)(\bar{\rho}-\varepsilon_1)\ln r_n/\rho \ln R}$$
$$= C_\varepsilon^{(1)} + (n-1) r_n^{(\bar{\rho}_n-\varepsilon_1)(1+\varepsilon/\rho)}, \tag{2.6.8}$$

$$I_2 \leqq \exp\left\{\frac{(\bar\rho_n - \varepsilon_1)(\rho+\varepsilon)\ln r_n}{\frac{C(R)}{\ln r_n}(\bar\rho_n - \varepsilon_1) + \frac{\rho(\ln r_n - C(R))}{\ln r_n}}\right\} = \exp\left\{\frac{(\bar\rho_n - \varepsilon_1)(\rho+\varepsilon)\ln^2 r_n}{(\bar\rho_n - \varepsilon_1 - \rho)C(R) + \rho \ln r_n}\right\}$$

$$= \exp\{(\bar\rho_n - \varepsilon_1)(1+\varepsilon/\rho)(1+o(1))\ln r_n\} = r_n^{(\bar\rho_n-\varepsilon_1)(1+\varepsilon/\rho)(1+o(1))}, \tag{2.6.9}$$

where $o(1) \to 0$ as $r_n \to \infty$, and R is an arbitrary fixed number.

We now show that if $r_0 = e^{\bar\rho_n/\rho}$, inequality (2.6.7) provides an estimate for II. It will clearly suffice to show that for any fixed $R > e^{\bar\rho_n/\rho}$ and all sufficiently large r_n the inequalities $s_i \leqq e^{\bar\rho_n/\rho}$, $i = 1, \cdots, n-1$, hold. We have

$$s_i = \exp\left\{\ln R \cdot \frac{1}{\lambda\alpha_i/\beta_i + \mu}\right\} = \exp\left\{\frac{\ln R \ln r_n}{C(R)(\ln r_n - C(R)) + \ln r_n - C(R)}\right\}$$

$$= \exp\left\{\frac{\ln R \ln r_n}{(\ln r_n - C(R))(C(R)+1)}\right\} = \exp\left\{\frac{(\bar\rho_n - \varepsilon_1)\ln r_n}{\rho(\ln r_n - C(R))}\right\} \leqq e^{\bar\rho_n\rho}.$$

The last inequality holds for every fixed $R \geqq e^{\bar\rho_n/\rho}$ and all sufficiently large r_n; hence for those values of R and r_n we can estimate II by (2.6.7). We get

$$\begin{aligned}
II &\leqq C_{\varepsilon_2}^{(2)} + (\gamma+\varepsilon_2)\exp\left\{\frac{\bar\rho_n\beta_n \ln r_n}{\lambda\alpha_n + \mu\beta_n}\right\} \\
&\leqq C_{\varepsilon_2}^{(2)} + (\gamma+\varepsilon_2)\exp\left\{\bar\rho_n \ln r_n + \bar\rho_n \ln r_n\left(\frac{\beta_n}{\lambda\alpha_n+\mu\beta_n} - 1\right)\right\} \\
&= C_{\varepsilon_2}^{(2)} + (\gamma+\varepsilon_2) r_n^{\bar\rho_n}\exp\left\{\bar\rho_n \frac{\lambda(\beta_n - \alpha_n)}{\lambda\alpha_n + \mu\beta_n}\ln r_n\right\} \\
&= C_{\varepsilon_2}^{(2)} + (\gamma+\varepsilon_2) r_n^{\bar\rho_n}\exp\left\{\frac{\bar\rho_n C(R)(\bar\rho - \rho_n + \varepsilon_1)}{(\rho_n - \varepsilon_1)\frac{C(R)}{\ln r_n} + \rho\frac{\ln r_n - C(R)}{\ln r_n}}\right\} \\
&= C_{\varepsilon_2}^{(2)} + (\gamma+\varepsilon_2) r_n^{\bar\rho_n}\exp\left\{\frac{(\rho - \bar\rho_n + \varepsilon_1)\bar\rho_n C(R)}{(1+o(1))\rho}\right\},
\end{aligned} \tag{2.6.10}$$

where $o(1) \to 0$ as $r_n \to \infty$, for any fixed R.

We have not yet chosen the numbers $\varepsilon > 0$, $\varepsilon_1 \in (0, \bar\rho_n)$ and $\varepsilon_2 > 0$. Take ε so small that

$$(\bar\rho_n - \varepsilon_1)\left(1 + \frac{\varepsilon}{\rho}\right) < \bar\rho_n - \frac{\varepsilon_1}{2}.$$

It then follows from inequalities (2.6.8) and (2.6.9) that $I \leqq r_n^{\bar\rho_n - \varepsilon_1/4}$ for any fixed $R > e^{\bar\rho_n/\rho}$ and all sufficiently large r_n. Hence, by (2.6.10),

$$\sigma_n(R, \cdots, R) = \varlimsup_{r_n\to\infty}\frac{\Phi(R, \cdots, R, r_n)}{r_n^{\bar\rho_n}} \leqq \varlimsup_{r_n\to\infty}(\gamma+\varepsilon_2)\exp\left\{\frac{(\rho - \bar\rho_n + \varepsilon_1)\bar\rho_n C(R)}{\rho(1+o(1))}\right\}$$

$$= (\gamma+\varepsilon_2) R^{(\rho-\bar\rho_n+\varepsilon_1)\bar\rho_n/(\bar\rho_n-\varepsilon_1)} e^{-(\rho-\bar\rho_n+\varepsilon_1)\bar\rho_n/\rho}.$$

This inequality is valid for arbitrary ε_1 and ε_2, and since $\sigma_n(r_1, \cdots, r_{n-1})$ is independent of ε_1 and ε_2, the following inequality will also hold for $R \geqq e^{\bar{\rho}_n/\rho}$:

$$\sigma_n(R, \cdots, R) \leqq \gamma R^{\rho-\bar{\rho}_n} e^{-(\rho-\bar{\rho}_n)\bar{\rho}_n/\rho} = \gamma(Re^{-\bar{\rho}_n/\rho})^{\rho-\bar{\rho}_n}.$$

Our theorem is thus proved, on the assumption that $r_0 = e^{\bar{\rho}_n/\rho}$. The general case may be reduced to this by the substitution $r_i = r_i' r_i^0 e^{-\bar{\rho}_n/\rho}$. One then gets the following inequality, which is valid for all $R \geqq r_0$, $R = R' r_0 e^{-\bar{\rho}_n/\rho}$:

$$\begin{aligned}\sigma_n(R, \cdots, R) &\leqq \sigma_n(R' r_1^0 e^{-\bar{\rho}_n/\rho}, \cdots, R' r_{n-1}^0 e^{-\bar{\rho}_n/\rho}) \\ &\leqq \gamma(R'/e^{\bar{\rho}_n/\rho})^{\rho-\bar{\rho}_n} = \gamma(R/r_0)^{\rho-\bar{\rho}_n}.\end{aligned}$$

This completes the proof.

6. *On convergence of the integral* $\int^{\infty} \Phi(r_1, \cdots, r_n)\, r_n^{-\alpha}\, dr_n$. The convergence or divergence of the integral

$$\int^{\infty} \frac{\Phi(r)}{r_n^{\alpha+1}}\, dr_n \tag{2.6.11}$$

is a useful characteristic of the growth of a function $\Phi(r) \in \mathfrak{A}$ in many theoretical questions; prominent among these is the proof of analogs of the Weierstrass factorization theorem in C^n. From the properties of such integrals, considered in § 4 of the Introduction, it follows that the integral (2.6.11) converges for $\alpha > \bar{\rho}_n$ and diverges for $\alpha < \bar{\rho}_n$. In the case $\alpha = \bar{\rho}_n$, we have

THEOREM 2.6.6. *Let* $\Phi(r) \in \mathfrak{A}$ *be a function of order* $\rho(\Phi) = \rho < \infty$, *and order* $\bar{\rho}_n > 0$ *in the variable* r_n. *If the integral*

$$\int^{\infty} \frac{\Phi(r)}{r_n^{\bar{\rho}_n+1}}\, dr_n$$

converges for some values $r_1 = r_1^0, \cdots, r_{n-1} = r_{n-1}^0$ *then it converges for any* $r_1, \cdots, r_{n-1}$.

PROOF. To simplify the notation, we set

$$\Phi(r_1^0, \cdots, r_{n-1}^0, r_n) = \varphi(r_n)\, r_n^{\bar{\rho}_n}.$$

Duplicating the proof of Theorem 2.6.5 up to inequality (2.6.10), we see that for any fixed R and all sufficiently large r_n

$$\Phi(R, \cdots, R, r_n) \leqq r_n^{\bar{\rho}_n-\varepsilon} + \varphi\left(r_n^{\frac{\beta_n}{\lambda\alpha_n+\mu\beta_n}}\right) r_n^{\frac{\bar{\rho}_n\beta_n}{\lambda\alpha_n+\mu\beta_n}} \leqq r_n^{\bar{\rho}_n-\varepsilon} + D_R\varphi\left(r_n^{\frac{\beta_n}{\lambda\alpha_n+\mu\beta_n}}\right) r_n^{\bar{\rho}_n}.$$

Here D_R is a constant depending only on R.

Since

$$\int^{\infty} \frac{r_n^{\bar{\rho}_n-\varepsilon}}{r_n^{\bar{\rho}_n+1}}\, dr_n < \infty,$$

it follows that, in order to prove the theorem for $r_1 = \cdots = r_{n-1} = R$, it will suffice to prove convergence of the integral

$$\int_1^{\infty} \varphi\left(r_n^{\beta_n/(\lambda\alpha_n+\mu\beta_n)}\right) r_n^{-1}\, dr_n.$$

To this end, we perform the following substitution of variables:

$$t = r_n^{\beta_n/(\lambda\alpha_n+\mu\beta_n)} = e^{\ln^2 r_n/(\ln r_n+K)},$$

where

$$K = -\frac{\rho - \bar{\rho}_n + \varepsilon_1}{\rho} C(R).$$

The result is

$$\int_1^{\infty} \varphi\left(r_n^{\beta_n/(\lambda\alpha_n+\mu\beta_n)}\right) r_n^{-1}\, dr_n = \frac{1}{2}\int_1^{\infty} \frac{\varphi(t)}{t}\left(1 + \frac{\ln t + 2K}{\sqrt{\ln^2 t + 4K\ln t}}\right) dt.$$

That the last integral is convergent follows immediately from the assumptions of the theorem, according to which the integral $\int_1^{\infty} t^{-1}\varphi(t)\, dt$ is convergent. Thus for $r_1 = \cdots = r_{n-1} = R$ the proof is complete. The truth of the theorem for arbitrary $r_1, \cdots, r_{n-1}$ now follows from the fact that the function $\Phi(r_1, \cdots, r_n)$ is monotone nondecreasing in each of its variables. This completes the proof.

§ 7. Functions of class $\mathfrak{B}$

1. *Definition of the class* $\mathfrak{B}$. *Relation with functions of class* $\mathfrak{A}$. Let $\mathfrak{B}$ denote the class of functions $\Phi(z, t)$, defined for $t \geqq 0$, $z \in C^n$, such that the functions $\Phi(z, |w|)$, $w \in C^1$, are plurisubharmonic in C^{n+1}, and the functions $e^{\Phi(z,t)}$ continuous in $C^n \times R_+^1$.[28] Examples of functions in this class are such important functions as $\ln M_f(z, t)$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(z, te^{i\theta})\right| d\theta,$$

where $f(z, w)$ is an entire function of the variables $z_1, \cdots, z_n$ and w, and

$$M_f(z, t) = \max_{|w|=t} \left|f(z, \omega)\right|.$$

[28] Functions of class $\mathfrak{B}$ were introduced in Ronkin [11], where the principal results of this section were presented without proof. Note that the definition of $\mathfrak{B}$ given in [11] is inadequate, in that the continuity of the function $\exp\{\Phi(z, t)\}$, which is essential for the proof of our results, was not required there.

Note that for fixed z any function $\Phi(z,t)$, as a function of t, is a continuous and monotone nondecreasing function of t. This follows from the already established fact that the same property holds for functions of class $\mathfrak{A}$ in C^1. A more precise result, which does the same for investigation of functions in class $\mathfrak{B}$ as inequality (2.6.1) for functions of class $\mathfrak{A}$, is the estimate furnished by the following lemma.

LEMMA 2.7.1. *Let E be a closed plane subset of the disk $|z_1| < R$, of positive capacity, E^* the component of the complement of E containing the point at infinity, and $\omega(z_1)$ the harmonic measure of E relative to the disk $|z_1| < R$. Let the function $\Phi(z_1, t) \in \mathfrak{B}$ satisfy the following conditions*:

(1) $\Phi(z_1, T_1) \leqq A \ \forall z_1 \in \partial E^*$,

(2) $\Phi(z_1, T_2) \leqq B$ *for* $|z_1| = R$,

with certain constants T_1, T_2, A, B. Then, throughout the domain $G_R(E) = E^* \cap \{z_1 : |z_1| < R\}$

$$\Phi(z_1, T_1^{\omega(z_1)} T_2^{1-\omega(z_1)}) \leqq A\omega(z_1) + B(1 - \omega(z_1)). \tag{2.7.1}$$

PROOF. We denote

$$\varphi(z_1) = \Phi(z_1, T_1^{\omega(z_1)} T_2^{1-\omega(z_1)})$$

and show that the function $\varphi(z_1)$ is subharmonic in the domain $G_R(E)$. To do this, choose an arbitrary disk $\mathscr{C}(z_1^0, r) = \{z_1 : |z_1 - z_1^0| < r\}$ in the domain $G_R(E)$, and let $h(z_1)$ be the conjugate harmonic function of $\omega(z_1)$ in this disk. Denote the function $\omega(z_1) + ih(z_1)$, which is analytic in the disk $\mathscr{C}(z_1^0, r)$, by $g(z_1)$. Since $\Phi(z, t) \in \mathfrak{B}$, the function $\Phi(z_1, |z_2|)$ is plurisubharmonic in C^2. Hence by Corollary 1 to Theorem 2.1.1 it follows that the function

$$\Phi(z_1, |\exp\{g(z_1) \ln T_1 + (1 - g(z_1)) \ln T_2\}|)$$

is subharmonic in the disk $\mathscr{C}(z_1^0, r)$. But

$$|\exp\{g(z_1) \ln T_1 + (1 - g(z_1)) \ln T_2\}| = T_1^{\omega(z_1)} T_2^{1-\omega(z_1)}$$

and hence

$$\Phi(z_1, |\exp\{g(z_1) \ln T_1 + (1 - g(z_1)) \ln T_2\}|) = \varphi(z_1).$$

Consequently the function $\varphi(z_1)$ is subharmonic in every disk $\mathscr{C}(z_1^0, r) \subset G_R(E)$, and thus in the entire domain $G_R(E)$. It readily follows from condition (1) of the lemma and from the definition of harmonic measure that at any point $z_1 \in \{z_1 : |z_1| = R\}$ we have the inequality

$$\overline{\lim_{\substack{z_1' \to z_1 \\ z_1' \in G_R(E)}}} \varphi(z_1') \leqq B. \tag{2.7.2}$$

Similarly, at all points of the set ∂E^*, except perhaps for a set of zero capacity,

$$\overline{\lim_{\substack{z_1' \to z_1 \\ z_1' \in G_R(E)}}} \varphi(z_1') \leqq A. \tag{2.7.3}$$

Since the function $\varphi(z_1)$ is subharmonic in the domain $G_R(E)$, it follows from (2.7.1) and (2.7.2), by (1.4.5), that for $z_1 \in G_R(E)$

$$\varphi(z_1) \leqq A\omega(z_1) + B(1 - \omega(z_1)).$$

This proves the lemma.

We are interested in the growth of the functions of class $\mathfrak{B}$ as functions of t. Some assertions to this effect follow as an easy corollary of the results in § 6. The reason for this is that the function

$$M_\Phi(r, t) = M_\Phi(r_1, \cdots, r_n, t) = \max_{\substack{|z_i| = r_i \\ i=1,\cdots,n}} \Phi(z, t),$$

whose growth is bound up with that of the function $\Phi(z, t) \in \mathfrak{B}$, is easily seen to belong to class $\mathfrak{A}$. Indeed, $M_\Phi(|z_1|, \ldots, |z_n|, |w|)$ is obviously upper semicontinuous, and

$$M_\Phi(|z_1|, \cdots, |z_n|, |w|) = \max_{\substack{0 \leqq \theta \leqq 2\pi \\ i=1,\cdots,n}} \Phi(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n}, |w|).$$

By the definition of the class $\mathfrak{B}$, the functions $\Phi(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n}, |w|)$ are plurisubharmonic for any $\theta_1, \cdots, \theta_n$. Consequently (see § 1 of Chapter 1) $M_\Phi(|z_1|, \cdots, |z_n|, |w|)$ is also plurisubharmonic, and therefore $M_\Phi(r, t) \in \mathfrak{A}$.

As already mentioned, the construction of the function $M_\Phi(r, t)$ is such that its various growth characteristics as a function of class $\mathfrak{A}$ are natural candidates for the analogous roles with respect to the function $\Phi(z, t)$. Thus the *order of the function* $\Phi(z, t) \in \mathfrak{B}$ is defined as the order $\rho(M_\Phi)$ of the function $M_\Phi(r, t) \in \mathfrak{A}$, and the *upper order* and *upper type of the function* $\Phi(z, t)$ *in the variable* t are defined as the order $\bar{\rho}_t(M_\Phi)$ and upper type $\bar{\sigma}_t(M_\Phi)$ of the function $M_\Phi(r, t)$ in the variable t.

The growth characteristics of the function $\Phi(z, t)$ thus defined of course possess all the properties established in § 6.

Growth characteristics of the function $\Phi(z, t)$ in the variable t, more delicate than $\bar{\sigma}_t$, $\bar{\rho}_t$ or

$$\sigma_t(r; M_\Phi, \bar{\rho}_t) = \overline{\lim_{t \to \infty}} \frac{M_\Phi(r, t)}{t^{\bar{\rho}_t}},$$

can be obtained by investigating this growth directly, i.e. without recourse to the function $M_\Phi(r, t)$. When this is done, the basic characteristics are again order and type.

2. *Order of the function* $\Phi(z, t)$ *in the variable* t. Let $\Phi(z, t) \in \mathfrak{B}$. Denote by $\rho(z; \Phi)$ the order of the function $\Phi(z, t)$ in t for fixed z; in other words, define the function $\rho(z; \Phi)$ by

$$\rho(z; \Phi) = \overline{\lim_{t\to\infty}} \frac{\ln \Phi^+(z, t)}{\ln t},$$

where, as usual, $\Phi^+ = \max [0, \Phi]$. It is clear that

$$\rho(z; \Phi) \leqq \bar{\rho}_t(M_\Phi). \tag{2.7.4}$$

Moreover, it can be shown that for all z, except possibly for a set which is in a certain sense meager, equality must hold in (2.7.4). To be precise:

THEOREM 2.7.1. *Let $\Phi(z, t) \in \mathfrak{B}$ and assume that $\rho(z; \Phi) < \infty$ on some set E of positive Γ-capacity. Then the upper order $\bar{\rho}_t(M_\Phi)$ of the function $\Phi(z, t)$ in the variable t is finite, and the equality $\rho(z; \Phi) = \bar{\rho}_t(M_\Phi)$ holds everywhere in C^n, except perhaps for K_Φ in class F_σ which satisfies the following condition*:

(A) *$K_\Phi \neq C^n$, and the intersection of K_Φ with any analytic plane $\{z: z_i = a_i w + b_i, i = 1, \cdots, n\}$ which is not entirely in K_Φ has zero capacity.*[29]

PROOF. For each $m = 1,2, \cdots$, let E_m be the set of all points z such that

$$\sup_{t\geqq m} \frac{\Phi^+(z, t)}{t^m} \leqq 1.$$

Obviously every set E_m is closed. Observe, moreover, that for any point $z^0 \in E$ there exists $m = m(z^0)$ such that $z^0 \in E_m$. Indeed, let $z^0 \in E$. Then $\rho(z^0; \Phi) < \infty$, and hence there exists an integer m_1 such that $\rho(z^0; \Phi) < m_1$. Consequently there exists a t_0 such that for all $t > t_0$

$$\Phi^+(z^0, t) \leqq t^{m_1}. \tag{2.7.5}$$

Take some integer $m_0 > \max [m_1, t_0]$. It then follows from (2.7.5) that $\Phi^+(z, t) \leqq t^{m_0}$ for any $t > m_0$. Consequently $z^0 \in E_{m_0}$. We have thus proved that $E \subset \bigcup_1^\infty E_m$. By assumption, Γ-Cap $E > 0$. Therefore

$$\Gamma\text{-Cap} \bigcup_{m=1}^{\infty} E_m > 0. \tag{2.7.6}$$

Now note that, by Lemma 2.2.4, any Γ-projection $\Gamma_n^0(E_m)$ is an F_σ set, and therefore (see Lemma 2.2.3)

$$\Gamma_n^1\Big(\bigcup_{m=1}^{\infty} E_m\Big) = \bigcup_{m=1}^{\infty} \Gamma_n^1(E_m).$$

Similarly, if φ is a unitary transformation of C^n, then

$$\Gamma_n^1\Big(\varphi\Big(\bigcup_{m=1}^{\infty} E_m\Big)\Big) = \bigcup_{m=1}^{\infty} \Gamma_n^1(\varphi E_m). \tag{2.7.7}$$

[29] As we remarked in § 2, sets satisfying condition (A) have zero Γ-capacity.

By the definition of Γ-capacity it follows from (2.7.6) that there is at least one unitary transformation φ such that

$$\operatorname{Cap}_2 \Gamma_n^1\Big(\varphi\Big(\bigcup_{m=1}^{\infty} E_m\Big)\Big) > 0.$$

Hence by (2.7.7) we conclude that for some $m = m'$

$$\operatorname{Cap}_2 \Gamma_n^1(\varphi E_{m'}) > 0. \tag{2.7.8}$$

Without loss of generality we may assume that φ is the identity transformation, and thus instead of (2.7.8) we have the inequality

$$\operatorname{Cap}_2 \Gamma_n^1(E_{m'}) > 0.$$

Therefore (see Lemma 2.2.5) there exist positive numbers $\alpha_1, \cdots, \alpha_n$ and a system of bounded plane sets $G_1, G_2(z_1), \cdots, G_n(z_1, \cdots, z_{n-1})$ such that the sets generated by the recursion relations

$$\begin{aligned}
E_1' &= G_1,\\
E_2' &= \{(z_1, z_2): z_1 \in E_1', z_2 \in G_2(z_1)\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
E_n' &= \{(z_1, \cdots, z_n): (z_1, \cdots, z_{n-1}) \in E_{n-1}', z_n \in G_n(z_1, \cdots, z_{n-1})\}
\end{aligned}$$

satisfy the conditions

(a) $\operatorname{Cap}_2 G_l(z_1, \cdots, z_{l-1}) \geqq \alpha_l \ \forall (z_1, \cdots, z_{l-1}) \in E_{l-1}'$, $l = 2,3, \cdots, n$,

(b) $\operatorname{Cap}_2 G_1 = \alpha_1 > 0$,

(c) $E_n' \subset E_{m'}$.

Without loss of generality we may assume that the set $E_{m'}$ is bounded, and thus lies in some polydisk $\{z: |z_i| < R_0, i = 1, \cdots, n\}$. Thus all sets G_l taking part in the construction of E_n' lie in a disk of radius R_0 about the origin. Moreover, since $E_n' \subset E_{m'}$, it follows that for $t \geqq m'$ and $z \in E_n'$

$$\Phi^+(z, t) \leqq t^{m'}. \tag{2.7.9}$$

As a function of z_n and t for fixed $z_1, \cdots, z_{n-1}$, the function $\Phi^+(z, t)$ is obviously in class $\mathfrak{B}$. Let us estimate it, using Lemma 2.7.1.

Let $t_1 > t_2 > m'$, and let G_n^* be the component of the complement of the set $G_n(z_1, \cdots, z_{n-1})$ that contains the point at infinity, and $\omega(z_n) = \omega_{R_n}(z_n; G)$ the harmonic measure of G_n relative to the disk $\{z_n: |z_n| < R_n\}$, where $R_n > R_0$. By Lemma 2.7.1, for $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$ and $z_n \in \{z_n: |z_n| < R_n\} \cap G_n^*$ we have

$$\begin{aligned}
\Phi^+(z, t_1^{\omega} t_2^{1-\omega}) &\leqq \omega(z_n) \max_{z_n \in G_n} \Phi^+(z, t_1) + (1 - \omega(z_n)) \max_{|z_n| = R_n} \Phi^+(z, t_2)\\
&\leqq t_1^{m'} + M_{\Phi^+}(R_1, \cdots, R_n, t_2),
\end{aligned} \tag{2.7.10}$$

where $R_1, \cdots, R_{n-1}$, like R_n, are arbitrary positive numbers such that $R_i > R_0$, $i = 1, \cdots, n$. Given numbers $R_i' > R_0$ and $\varepsilon > 0$, take numbers R_i such that

$$\frac{\ln (R_i' + R_0) - \ln \alpha_i}{\ln (R_i - R_0) - \ln R_0} \leqq \varepsilon \qquad (i = 1, \cdots, n). \tag{2.7.11}$$

Then, by the estimate of harmonic measure given in Lemma 1.4.1, it follows that $t_1^{\omega(z_n)} \geqq t_1^{1-\varepsilon}$ for $|z_n| = R_n'$. Since the function $\Phi^+(z, t)$ is monotone in t, it follows from this inequality and from (2.7.10) that for $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$ and $|z_n| \leqq R_n'$

$$\Phi^+(z, t_1^{1-\varepsilon}) \leqq t_1^{m'} + M_{\Phi^+}(R_1, \cdots, R_n, t_2).$$

Denoting $t_1^{1-\varepsilon}$ by t and taking into account that the number t_1 is arbitrary, we see that for $t > t_2$ and $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$ the function

$$M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t) = \max_{|z_n| = r_n} \Phi^+(z, t)$$

satisfies the condition

$$M_{\Phi^+}(z_1, \cdots, z_{n-1}, R_n', t) \leqq t^{m'/(1-\varepsilon)} + M_{\Phi^+}(R_1, \cdots, R_n, t_2). \tag{2.7.12}$$

It is readily seen that for fixed r_n the function $M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t)$ is in the class $\mathfrak{B}$ as a function of the variables t and $z_1, \cdots, z_{n-1}$. Now the structure of the set E_{n-1}' is the same as that of E_n'. Thus $M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t)$ satisfies an estimate similar to that of $\Phi(z, t)$, with the sole difference that instead of inequality (2.7.9) we must use (2.7.12). The final result is that for all $t > t_2$ and $(z_1, \cdots, z_{n-2}) \in E_{n-2}'$ the function

$$M_{\Phi^+}(z_1, \cdots, z_{n-2}, r_{n-1}, r_n, t) = \max_{\substack{|z_{n-1}| = r_{n-1} \\ |z_n| = r_n}} \Phi^+(z, t)$$

satisfies the condition

$$M_{\Phi^+}(z_1, \cdots, z_{n-2}, R_{n-1}', R_n', t) \leqq t^{m/(1-\varepsilon)^2} + 2M_{\Phi^+}(R_1, \cdots, R_n, t_2).$$

Repeating these arguments another $n-2$ times, we get the following estimate, valid for $t > t_2$ and arbitrary $R_1' > R_0, \cdots, R_n' > R_0$:

$$M_{\Phi^+}(R_1', \cdots, R_n', t) \leqq t^{m'/(1-\varepsilon)^n} + nM_{\Phi^+}(R_1, \cdots, R_n, t_2), \tag{2.7.13}$$

where the numbers $R_1, \cdots, R_n$ are defined in terms of the given numbers $R_1', \cdots, R_n'$ by (2.7.11). It follows from this estimate that the order of the function $M_{\Phi^+}(R_1', \cdots, R_n', t)$ in the variable t is finite for any fixed $R_1', \cdots, R_n'$, and, moreover, since $\varepsilon > 0$ is arbitrary, does not exceed m'. Thus the upper order $\bar{\rho}_t(M_\Phi)$ is finite, and $\bar{\rho}_t(M_\Phi) \leqq m'$.

Now consider the set

$$K_\Phi = \{z: \rho(z; \Phi) < \bar{\rho}_t(M_\Phi)\}.$$

This set may also be expressed as

$$K_\Phi = \left\{z: \overline{\lim_{t \to \infty}} \frac{\ln \Phi^+(z, t)}{\ln t} < \bar{\rho}_t(M_\Phi)\right\},$$

whence it immediately follows (see § 5 of Chapter 1) that $K_\Phi \in F_\sigma$. To prove that this set satisfies condition (A), we observe first that, if we replace the sets E_m in the foregoing constructions by the sets

$$E_{m,\gamma} = \{z: \Phi(z, t) \leqq t^{\gamma - 1/m} \ \forall t \geqq m\}$$

we get the following lemma.

LEMMA 2.7.2. *If the inequality $\rho(z; \Phi) < \gamma$ holds for some $\gamma > 0$ on a set of positive Γ-capacity, then $\bar{\rho}_t(M_\Phi) < \gamma$.*

It follows from this lemma that the set K_Φ has zero Γ-capacity. Indeed, were the Γ-capacity of K_Φ positive, it would follow from Lemma 2.7.2 (with $\gamma = \bar{\rho}_t(M_\Phi)$) that $\bar{\rho}_t(M_\Phi) < \bar{\rho}_t(M_\Phi)$, which is absurd. Thus Γ-Cap $K_\Phi = 0$, and so $K_\Phi \neq C^n$.

In the space C^1, Γ-capacity coincides with ordinary capacity. Hence $\operatorname{Cap}_2 K_\Phi = 0$ for $n = 1$, and thus the set K_Φ satisfies condition (A) in this case. If $n > 1$, consider the function

$$\varphi(w, t) = \Phi(a_1 w + b_1, \cdots, a_n w + b_n, t)$$

with arbitrary complex a_i and b_i. If the upper order $\bar{\rho}_t(M_\varphi)$ of this function satisfies the inequality $\bar{\rho}_t(M_\varphi) < \bar{\rho}_t(M_\Phi)$, then all points of the plane

$$L_{a,b} = \{z: z_i = a_i w + b_i, \ i = 1, \cdots, n, \ w \in C^1\}$$

belong to the set K_Φ, i.e. $L_{a,b} \subset K_\Phi$. But if $\bar{\rho}_t(M_\varphi) = \bar{\rho}_t(M_\Phi)$, then the intersection $K_\Phi \cap L_{a,b}$ is clearly the subset of $L_{a,b}$ defined by the points w in the set

$$K_\varphi = \{w: \rho(w; \varphi) < \bar{\rho}_t(M_\Phi)\}.$$

But, as we proved above, $\operatorname{Cap}_2 K_\varphi = 0$. Consequently $\operatorname{Cap}_2 \{K_\Phi \cap L_{a,b}\} = 0$. Combined with the foregoing, this proves that the set K_Φ satisfies condition (A). Q.E.D.

3. *Type of the function $\Phi(z, t)$ in the variable t.* Let $\Phi(z, t) \in \mathfrak{B}$. Denote by $\sigma_\Phi(z) = \sigma(z; \Phi, \rho)$ the type of the function $\Phi(z, t)$ for order ρ:

$$\sigma_\Phi(z) = \overline{\lim_{t \to \infty}} \frac{\Phi(z, t)}{t^\rho}.$$

As usual, $\sigma_\Phi^*(z)$ will denote the regularization of $\sigma_\Phi(z)$.

THEOREM 2.7.2. *Let $\Phi(z, t) \in \mathfrak{B}$, and let $r_1^0, \cdots, r_n^0 > 0$ be such that $\sigma_t(r^0; M_\Phi, \rho) < \infty$. Then the function $\sigma_\Phi^*(z)$ is plurisubharmonic in the polydisk $\mathscr{E}_0 = \{z: |z_i| < r_i^0, \ i = 1, \cdots, n\}$, the equality*

$$\sigma_t(r; M_\Phi, \rho) = \sup_{\substack{|z_i| < r_i \\ i = 1, \cdots, n}} \sigma_\Phi(z) \tag{2.7.14}$$

holds for arbitrary $r_1 < r_1^0, \cdots, r_n < r_n^0$, *and the set* $K_{\Phi,\sigma}$ *of all points* $z \in \mathscr{E}_0$ *such that* $\sigma_\Phi(z) < \sigma_\Phi^*(z)$ *is an* F_σ *set and has zero* Γ*-capacity.*

PROOF. Since $\sigma_t(r^0; M_\Phi, \rho) < \infty$, it follows that for some $\gamma < \infty$ and all sufficiently large t (say, from $t = t_0$ on),

$$M_\Phi(r^0, t)/t^\rho \leqq \gamma.$$

Consequently the family $\mathfrak{F}$ of functions $\{\Phi(z, t)/t^\rho\}_{t \geqq t_0}$ is uniformly bounded in the polycylinder $\mathscr{E}_0$. The function $\Phi(z, t)$, as follows from the definition of the class $\mathfrak{B}$, is a plurisubharmonic function of $z_1, \ldots, z_n$ for any t. Consequently (§ 3) the function $\sigma_\Phi^*(z)$, as the regularization of

$$\sigma_\Phi(z) = \overline{\lim_{t \to \infty}} \frac{\Phi(z, t)}{t^\rho},$$

is plurisubharmonic in the polydisk $\mathscr{E}_0$. Further, it is obvious that the set $K_{\Phi,\sigma}$ is exactly the set $S_\mathfrak{F}$ for this family $\mathfrak{F}$. Therefore (see § 3) $K_{\Phi,\sigma}$ is an F_σ set and Γ-Cap $K_{\Phi,\sigma} = 0$.

We now show that equality (2.7.14) holds for $r_1 < r_1^0, \cdots, r_n < r_n^0$. To do this we first observe that by Hartogs' Lemma (§ 1 of Chapter 1) the fact that the functions $t^{-\rho}\Phi(z, t)$ are uniformy bounded implies that for any $z \in \{z: |z_i| < r_i', i = 1, \cdots, n\}$, where $r_i' < r_i$, and any $\varepsilon > 0$ the inequality

$$\frac{\Phi(z, t)}{t^\rho} \leqq \sup_{\substack{|z_i| < r_i' \\ i=1,\cdots,n}} \sigma_\Phi^*(z) + \varepsilon$$

holds for all t from some $t = t_0(r', \varepsilon)$ on. Consequently

$$\frac{M_\Phi(r', t)}{t^\rho} \leqq \sup_{\substack{|z_i| \leqq r_i' \\ i=1,\cdots,n}} \sigma_\Phi^*(z) + \varepsilon,$$

whence it follows that

$$\sigma_t(r'; M_\Phi, \rho) \leqq \sup_{\substack{|z_i| \leqq r_i' \\ i=1,\cdots,n}} \sigma_\Phi^*(z) + \varepsilon. \tag{2.7.15}$$

As we remarked in § 6, the function $\sigma_t(r; M_\Phi, \rho)$ is a convex function of $\ln r_1, \cdots, \ln r_n$, hence also continuous. Hence, since the number $\varepsilon > 0$ is arbitrary, we conclude from (2.7.15) that

$$\sigma_t(r; M_\Phi, \rho) \leqq \sup_{\substack{|z_i| \leqq r_i \\ i=1,\cdots,n}} \sigma_\Phi^*(z). \tag{2.7.16}$$

The converse inequality is obvious. Consequently

$$\sigma_t(r; M_\Phi, \rho) = \sup_{\substack{|z_i| \leqq r_i \\ i=1,\cdots,n}} \sigma_\Phi^*(z). \qquad \text{Q.E.D.}$$

COROLLARY. *Let $\Phi(z, t) \in \mathfrak{B}$ and $\bar{\sigma}_t(M_\Phi) < \infty$. Then the set of all points z such that $\sigma_\Phi(z) < \bar{\sigma}_t(M_\Phi)$ is an F_σ set and has zero Γ-capacity.*

This statement follows easily from Theorem 2.7.2 for in this case $\bar{\sigma}_t(r; M_\Phi, \rho) \equiv \bar{\sigma}_t(M_\Phi)$ (see § 6.5).

If $\Phi(z, t)$ is a function of finite order $\rho(M_\Phi)$ such that $\sigma_t(r; M_\Phi, \rho) < \infty$ for some values $r_1 = r_1^0, \cdots, r_n = r_n^0$, it follows from Theorem 2.6.5 that this inequality is valid for arbitrary $r_1, \cdots, r_n$. The same assertion holds under weaker assumptions concerning the function $\Phi(z, t)$, as the following theorem shows.

THEOREM 2.7.3. *Let $\Phi(z, t) \in \mathfrak{B}$ be a function of finite order $\rho(M_\Phi) = \rho$. Assume that there exists $\gamma < \infty$ such that the inequality $\sigma(z; \Phi, \bar{\rho}) \leqq \gamma$ holds on some set $E \subset C^n$ of positive Γ-capacity, where $\bar{\rho} = \bar{\rho}_t(M_\Phi)$. Then for any $R \geqq 0$ and some constant C_Φ*

$$\sigma_t(R, \cdots, R; M_\Phi, \bar{\rho}) \leqq \gamma C_\Phi R^{\rho - \bar{\rho}}. \tag{2.7.17}$$

PROOF. Consider the sets

$$E_{m,\varepsilon} = \left\{z: \sup_{t \geqq m} \frac{\Phi(z, t)}{t^{\bar{\rho}}} \leqq \gamma - \frac{1}{m} + \varepsilon\right\}$$

and repeat the reasoning at the beginning of the proof of Theorem 2.7.1, replacing the sets E_m figuring there by the new sets $E_{m,\varepsilon}$. We get a system of plane closed sets G_1, $G_2(z_1)$, $\cdots$, $G_n(z_1, \cdots, z_{n-1})$ and corresponding sets $E_1', \cdots, E_n'$ such that for some $m_0 > 0$ we have

(a) $$\operatorname{Cap}_2 G_1 = \alpha_1 > 0,$$

(b) $$\inf_{E_{l-1}} \operatorname{Cap}_2 G_l(z_1, \cdots, z_{l-1}) = \alpha_l > 0, \quad l = 2, \cdots, n,$$

(c) $$\sup_{t \geqq m_0} \frac{\Phi^+(z, t)}{t^{\bar{\rho}}} \leqq \gamma + \varepsilon \qquad \forall z \in E_n',$$

(d) $$\max_{E_n'} |z| < \infty.$$

As already shown, for fixed $z_1, \cdots, z_{n-1}$ the function $\Phi(z, t)$ is in class $\mathfrak{B}$ as a function of z_n and t. Let us estimate it, using Lemma 2.7.1. It is obvious that inequality (2.7.10) holds here. Thus for $t_1 > t_2 > m_0$, $(z_1, \cdots, z_{n-1}) \in E_{n-1}$ and $z_n \in \{z_n: |z_n| < R_n\} \cap G_n^*$ we have

$$\begin{aligned}\Phi^+(z, t_1^\omega t_2^{1-\omega}) &\leqq \omega(z) \max_{z_n \in G_n} \Phi^+(z, t_1) + (1 - \omega(z)) \max_{|z_n| = R_n} \Phi^+(z, t_2) \\ &\leqq (\gamma + \varepsilon)\, t_1^{\bar{\rho}} + (1 - \omega(z_n))\, M_{\Phi^+}(|z_1|, \cdots, |z_{n-1}|, R_n, t_2),\end{aligned}$$

where, as before, $\omega(z_n) = \omega_{R_n}(z_n; G_n)$ denotes the harmonic measure of the set G_n relative to the disk $|z_n| < R_n$, G_n^* is the component of the complement of G_n

that contains the point at infinity, and R_n is an arbitrary number such that $R_n > 2R_0$, where $R_0 = \max_{z \in E_n'} |z|$. Setting $t_1^{\omega(z_n)} t_2^{1-\omega(z_n)} = t$ in this inequality and recalling that the order $\rho = \rho(M_\Phi)$ is finite, we see that for any $\varepsilon_1 > 0$ the inequality

$$\Phi(z, t) \leqq (\gamma + \varepsilon)(t/t_2)^{\bar{\rho}/\omega(z_n)} t_2^{\bar{\rho}} + C_{\varepsilon_1}^{(1)}(1 - \omega(z_n))\,(t_2^{\rho+\varepsilon_1} + |z_1|^{\rho+\varepsilon_1} + \cdots + |z_{n-1}|^{\rho+\varepsilon_1} + R_n^{\rho+\varepsilon_1}) + C_{\varepsilon_2}^{(2)}$$

holds for certain constants $C_{\varepsilon_1}^{(1)}$ and $C_{\varepsilon_2}^{(2)}$. Hence, by Lemma 1.4.1 it follows that for $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$, $R_0 < |z_n| < R_n$ and $t \geqq t_2$ we have

$$\begin{aligned} \Phi(z, t) \leqq (\gamma + \varepsilon)\, t_2^{\bar{\rho}} \exp\left\{\frac{\bar{\rho} \ln (t/t_2)}{1 - \dfrac{\ln(|z_n| + R_0) - \ln \alpha_n}{\ln (R_n - R_0) - \ln R_0}}\right\} \\ + C_{\varepsilon_1}^{(1)} \frac{\ln(|z_n| + R_0) - \ln \alpha_n}{\ln (R_n - R_0) - \ln R_0} \\ \cdot (t_2^{\rho+\varepsilon_1} + |z_1|^{\rho+\varepsilon_1} + \cdots + |z^{n-1}|^{\rho+\varepsilon_1} + R_n^{\rho+\varepsilon_1}) + C_{\varepsilon_1}^{(2)}. \end{aligned} \tag{2.7.18}$$

Hitherto the number $R_n > 2R_0$ has been arbitrary. We now take

$$R_n = t^{\bar{\rho}/(\rho+\varepsilon_1)} R_0 \qquad (t > 2^{(\rho+\varepsilon_1)/\rho}).$$

It then follows from (2.7.18) that for $r_n > R_0$

$$\begin{aligned} M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t) \leqq (\gamma + \varepsilon) t^{\bar{\rho}} \exp\left\{\frac{\bar{\rho}(\ln t - \ln t_2)(\ln (r_n + R_0) - \ln \alpha_n)}{\ln (t^{\bar{\rho}/\rho(+\varepsilon_1)} - 1) - \ln (r_n + R_0) + \ln \alpha_n}\right\} \\ + C_{\varepsilon_1}^{(1)} \frac{\ln (r_n + R_0) - \ln \alpha_n}{\ln (t^{\bar{\rho}/(\rho+\varepsilon_1)} - 1)} \{t_2^{\rho+\varepsilon_1} + |z_1|^{\rho+\varepsilon_1} + \cdots + |z_{n-1}|^{\rho+\varepsilon_1} + R_0^{\bar{\rho}} t^{\bar{\rho}}\} + C_{\varepsilon_1}^{(2)} \end{aligned} \tag{2.7.19}$$

Now take the number $t_0 > 2^{(\rho+\varepsilon_1)/\bar{\rho}}$ so that, for given $r_1, \cdots, r_n$ and all $t \geqq t_0$,

$$\exp\left\{\frac{\bar{\rho} \ln \dfrac{t}{t_2} \ln \dfrac{r_i + R_0}{\alpha_i}}{\ln \dfrac{t^{\bar{\rho}/(\rho+\varepsilon_1)} - 1}{r_i + R_0}}\right\} < \left(\frac{r_i + R_0}{\alpha_i}\right)^{\rho+\varepsilon_1} (1 + \eta)$$

and

$$C_{\varepsilon_1}^{(1)} \frac{\ln (r_i + R_0) - \ln \alpha_n}{\ln (t^{\bar{\rho}/(\rho+\varepsilon_1)} - 1)} R_0^{\rho+\varepsilon_1} < \eta.$$

This choice of t_0 is clearly possible. The estimate (2.7.19) now implies the following estimate for the function $M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t)$, valid for $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$, $t \geqq t_0$ and $r_n > R_0$:

$$\begin{aligned} M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t) \leqq t^{\bar{\rho}} \left\{(\gamma + \varepsilon)\left(\frac{r_n + R_0}{\alpha_n}\right)^{\rho+\varepsilon_1} (1 + \eta) + \eta\right\} \\ + C_{\varepsilon_1}^{(1)} \{t_2^{\rho+\varepsilon_1} + |z_1|^{\rho+\varepsilon_1} + \cdots + |z_{n-1}|^{\rho+\varepsilon_1}\} + C_{\varepsilon_1}^{(2)}. \end{aligned} \tag{2.7.20}$$

Now, as was shown above, $M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t)$ is in class $\mathfrak{B}$ for fixed r_n; moreover, the set E'_{n-1} has the same structure as the set E'_n. We can therefore repeat the above arguments and estimate the function

$$M_{\Phi^+}(z_1, \cdots, z_{n-2}, r_{n-1}, r_n, t) = \max_{|z_{n-1}|=r_{n-1}} M_{\Phi^+}(z_1, \cdots, z_{n-1}, r_n, t).$$

In so doing, we must replace the inequality

$$\Phi(z, t) \leqq (\gamma + \varepsilon)t^{\bar{\rho}} \qquad \forall t \geqq m_0,\ z \in E'_n,$$

by inequality (2.7.20). It then follows that, for $(z_1, \cdots, z_{n-2}) \in E'_{n-2}$, $r_{n-1} > R_0$, $r_n > R_0$ and $t > t_0$,

$$\begin{aligned} M_{\Phi^+}(z_1, \cdots, z_{n-2}, r_{n-1}, r_n, t) &\leqq \beta_2 t^{\bar{\rho}} \\ &+ C^{(1)}_{\varepsilon_1}(t_2^{\rho+\varepsilon_1} + |z_1|^{\rho+\varepsilon_1} + \cdots + |z_{n-2}|^{\rho+\varepsilon_1} + r_{n-1}^{\rho+\varepsilon_1}) \\ &+ C^{(1)}_{\varepsilon_1}(|z_1|^{\rho+\varepsilon_1} + \cdots + |z_{n-2}|^{\rho+\varepsilon_1} + r_n^{\rho+\varepsilon_1} + t_2^{\rho+\varepsilon_1}) + 2C^{(2)}_{\varepsilon_1}, \end{aligned}$$

where

$$\beta_2 = \left(\frac{r_{n-1} + R_0}{\alpha_{n-1}}\right)^{\rho+\varepsilon_1}\left\{(\gamma + \varepsilon)\left(\frac{r_n + R_0}{\alpha_n}\right)^{\rho+\varepsilon_1}(1 + \eta) + \eta\right\}(1 + \eta) + \eta.$$

Repeating the arguments another $n - 2$ times, we see that for sufficiently large t the following estimate holds:

$$M_{\Phi^+}(r_1, \cdots, r_n, t) \leqq \beta_n t^{\bar{\rho}} + nC^{(1)}_{\varepsilon_1}\left(t_2^{\rho+\varepsilon_1} + \sum_{i=1}^{n} r_i^{\rho+\varepsilon_1}\right) + nC^{(2)}_{\varepsilon_1}, \tag{2.7.21}$$

where β_n is defined by the recurrence relation

$$\beta_l = \beta_{l-1}\left(\frac{r_{n-l} + R_0}{\alpha_{n-l}}\right)^{\rho+\varepsilon_1}(1 + \eta) + \eta$$

with $\beta_0 = \gamma + \varepsilon$. This inequality immediately implies that

$$\sigma_t(r_1, \cdots, r_n;\ M_\Phi, \bar{\rho}) \leqq \beta_n. \tag{2.7.22}$$

Here the left-hand side is independent of the arbitrary positive numbers ε, η and ε_1. Thus inequality (2.7.22) remains valid for $\varepsilon_1 = \eta = \varepsilon = 0$. Hence for $r_1 > R_0, \cdots, r_n > R_0$ we have

$$\sigma_t(r;\ M_\Phi, \bar{\rho}) \leqq \gamma \prod_{j=1}^{n}\left(\frac{r_j + R_0}{\alpha_j}\right)^{\rho}.$$

The required inequality (2.7.17) now follows from Theorem 2.6.5.

4. *On convergence of the integral* $\int^{\infty} \Phi(z, t)t^{-\bar{\rho}_i-1}\,dt$. As we mentioned in the Introduction and recalled in § 6, the growth of a function $f(t)$ of finite order ρ and minimal type for order ρ may also be characterized by the convergence or

divergence of the integral $\int^{\infty} f(t)t^{-\rho-1}\,dt$. When we try to adopt this approach for functions $\Phi(z, t) \in \mathfrak{B}$, we must first observe that, as follows from Theorem 2.6.6, when $\rho(M_\Phi) < \infty$ the integral

$$\int^{\infty} \frac{M_\Phi(r, t)}{t^{\bar{\rho}_t+1}}\,dt$$

is either convergent for all $r_1 > 0, \cdots, r_n > 0$ or divergent for all such values. The following theorem makes this statement more precise.

THEOREM 2.7.4. *Let $\Phi(z, t) \in \mathfrak{B}$, and assume that the function $M_\Phi(r, t)$ has finite order ρ and positive upper order $\bar{\rho}_t = \bar{\rho}$ in the variable t. Let $K \subset C^n$ be a closed set of positive Γ-capacity, and define a function $\Phi(t; K)$ by*

$$\Phi(t; K) = \sup_{z \in K} \Phi(z, t).$$

If

$$\int^{\infty} \frac{\Phi(t; K)}{t^{\bar{\rho}+1}}\,dt < \infty,$$

then for any $r \in R_+^n$

$$\int^{\infty} \frac{M_\Phi(r, t)}{t^{\bar{\rho}+1}}\,dt < \infty.$$

PROOF. As before, we use Lemma 2.2.5 to construct from the set K sets G_1, $G_2(z_1)$, $\cdots$, $G_n(z_1, \cdots, z_{n-1})$ and corresponding sets E_1', ..., E_n' ($E_n' \subset K$). We then estimate the function $\Phi(z, t)$ just as in the proof of Theorem 2.7.3. Setting $t_2 = 1$, we then see that, for $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$, $R_0 < |z_n| < R_n$ and suitable constants $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(2)}$,

$$\Phi(z, t) \leqq \Phi^+(t^{1/\omega(z_n)}; K) + C_\varepsilon^{(1)}(1 + |z_1|^{\rho+\varepsilon} + \cdots + |z_{n-1}|^{\rho+\varepsilon} + R_n^{\rho+\varepsilon}) + C_\varepsilon^{(2)}. \quad (2.7.23)$$

Setting $R_n = t^{(\bar{\rho}-\varepsilon_1)/(\rho+\varepsilon)}R_0$, where $\varepsilon_1 \in (0, \bar{\rho})$, and adopting the abbreviated notation

$$\kappa = \frac{\ln (t^{(\bar{\rho}-\varepsilon_1)/(\rho+\varepsilon)} - 1)}{\ln (t^{(\bar{\rho}-\varepsilon_1)/(\rho+\varepsilon)} - 1) - \ln \dfrac{R_0 + r_n}{\alpha_n}},$$

we see from (2.7.23) that, for $(z_1, \cdots, z_{n-1}) \in E_{n-1}'$ and $r_n > R_0$,

$$M_\Phi(z_1, \cdots, z_{n-1}, r_n, t) \leqq \Phi^+(t^\kappa; K) + C_\varepsilon^{(1)}\left(1 + \sum_{i=1}^{n-1} |z_i|^{\rho+\varepsilon}\right) + C_\varepsilon^{(2)} + C_\varepsilon^{(1)} t^{\bar{\rho}-\varepsilon_1} R_0^{\rho+\varepsilon}.$$

Hence it follows that the function

$$M_\Phi(t; E'_{n-1}, r_n) = \sup_{E'_{n-1}} M_\Phi(z_1, \cdots, z_{n-1}, r_n, t)$$

satisfies the estimate

$$M_\Phi(t; E'_{n-1}, r_n) \leqq \Phi^+(t^\kappa; K) + C_\varepsilon^{(1)}(1 + (n-1)R_0^{\rho+\varepsilon}) + C_\varepsilon^{(2)} + C_\varepsilon^{(1)} t^{\bar{\rho}-\varepsilon_1} R_0^{\rho+\varepsilon},$$

and consequently

$$\int_1^\infty \frac{M_\Phi(t; E'_{n-1}, r_n)}{t^{\bar{\rho}+1}}\, dt \leqq \int_1^\infty \frac{\Phi^+(t^\kappa; K)}{t^{\bar{\rho}+1}}\, dt + \text{const.}$$

We claim that

$$\int^\infty \frac{\Phi^+(t^\kappa; K)}{t^{\bar{\rho}+1}}\, dt < \infty.$$

Indeed, it follows from the definition of κ that $k \leqq 1 + \alpha/\ln t$ for some constant $\alpha > 0$ and all sufficiently large t. Hence

$$\int^\infty \frac{\Phi^+(t^\kappa; K)}{t^{\bar{\rho}+1}}\, dt \leqq \int^\infty \frac{\Phi^+(t^{1+\alpha/\ln t}; K)}{t^{\bar{\rho}+1}}\, dt \leqq \int^\infty \frac{\Phi^+(e^\alpha t; K)}{t^{\bar{\rho}+1}}\, dt$$

$$= e^{-\alpha\bar{\rho}} \int^\infty \frac{\Phi^+(t; K)}{t^{\bar{\rho}+1}}\, dt < \infty.$$

Consequently

$$\int^\infty M_\Phi(t; E'_{n-1}, r_n)\, t^{-\bar{\rho}-1} < \infty.$$

Now repeat these arguments for the function $M_\Phi(z_1, \cdots, z_{n+1}, r_n, t)$, with the function $\Phi(t; K)$ replaced by $M_\Phi(t; E_{n-1}, r_n)$. It then follows that for any $r_{n-1} > R_0$ and $r_n > R_0$ the integral

$$\int^\infty \frac{M_\Phi(t; E_{n-2}, r_{n-1}, r_n)}{t^{\bar{\rho}+1}}\, dt,$$

where

$$M_\Phi(t; E_{n-2}, r_{n-1}, r_n) = \sup_{E_{n-2}} M_\Phi(z_1, \cdots, z_{n-2}, r_{n-1}, r_n, t),$$

is convergent. Repeating this procedure another $n-2$ times, we finally see that for any $r_1 > R_0, \cdots, r_n > R_0$

$$\int^\infty \frac{M_\Phi(r_1, \cdots, r_n, t)}{t^{\bar{\rho}+1}}\, dt < \infty.$$

Since the function $M_\Phi(r_1, \cdots, r_n, t)$ is monotone in the variables $r_1, \cdots, r_n$, this implies that the integral is convergent for arbitrary $r_1, \cdots, r_n$. Q.E.D.

CHAPTER 3

GROWTH OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

In this chapter we investigate the growth of entire functions of several complex variables, i.e. functions analytic throughout C^n. We shall make use of the growth characteristics defined in our study of the classes $\mathfrak{A}$ and $\mathfrak{B}$, as well as certain other characteristics.

§ 1. Associated orders, associated types and related concepts[1)]

Let $f(z)$ be an entire function in C^n. We denote

$$M_f(r) = M_f(r_1, \cdots, r_n) = \max_{\substack{|z_i|=r_i \\ i=1,\cdots,n}} |f(z)|.$$

As already mentioned (Chapter 2, §§ 1 and 6), the behavior and growth of the function $\ln^+ M_f(r) \in \mathfrak{A}$ can be described in terms of the characteristics introduced in our study of the class $\mathfrak{A}$. It is natural to define the growth characteristics of the function $f(z)$ to be simply those of $\ln^+ M_f(r)$. Thus the *order* of an entire function $f(z)$ (with respect to all its variables) is the order $\rho(\ln^+ M_f(r))$ of the function $\ln^+ M_f(r)$. To simplify the notation, we write $\rho(\ln^+ M_f) = \rho_f$. Similarly we define the *G-types, associated orders and associated types, orders and types in one of the variables* for the function $f(z)$. With these definitions, the properties of these characteristics established in the preceding chapter for functions of class $\mathfrak{A}$ in general will of course remain valid.

1. *Order and G-type of an entire function.* Let G be a complete domain in R_+^n. The *G-type $\sigma_{G,f}$ of an entire function $f(z)$ for order ρ_f* is defined to be the G-type $\sigma_G(\ln^+ M_f)$ of the function $\ln^+ M_f(r) \in \mathfrak{A}$ for order ρ_f, i.e.

$$\sigma_{G,f} = \overline{\lim_{t\to\infty}} \frac{\ln^+ M_G(t;f)}{t^{\rho_f}},$$

where $M_G(t;f) = \max_{r\in tG} M_f(r)$. For brevity we shall often replace ρ_f and $\sigma_{G,f}$ by ρ and σ_G, or simply σ.

[1)] Some generalizations of the concepts considered here may be found in Geče [1], [2], Maergoĭz [1]—[4], and Šeremeta [1].

THEOREM 3.1.1 (GOL'DBERG [1]).[2] *The order ρ and G-type σ_G of the entire function*

$$f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k$$

are related to the coefficients of its Taylor expansion by

$$\rho = \overline{\lim_{\|k\|\to\infty}} \frac{\|k\| \ln \|k\|}{-\ln|a_k|}, \tag{3.1.1}$$

$$(e\rho\sigma_G)^{1/\rho} = \overline{\lim_{\|k\|\to\infty}} \{\|k\|^{1/\rho}[|a_k| d_k(G)]^{1/\|k\|}\}, \tag{3.1.2}$$

where $d_k(G) = \max_{r\in G} r^k$.[3] *It is assumed in* (3.1.2) *that $\rho > 0$.*

PROOF. We first point out two elementary equalities which can be verified directly:

$$\min_{0<t<\infty} e^{t^\mu} t^{-m} = \left(\frac{e\mu\nu}{m}\right)^{m/\nu} \qquad (\mu > 0, \nu > 0, m \geqq 0), \tag{3.1.3}$$

$$\sup_{0<k<\infty} \left(\frac{a}{k}\right)^{k/\nu} t^k = \exp\left\{\frac{a}{e\nu} t^\nu\right\} \qquad (a > 0, \nu > 0). \tag{3.1.4}$$

Now consider the function $\varphi(z,w)$ of the variables $z_1,\cdots,z_n$ and w defined by

$$\varphi(z, w) = f(wz) = \sum_{\|k\|=0}^{\infty} a_k w^{\|k\|} z^k.$$

Set

$$\sum_{\|k\|=m} a_k z^k = P_m(z).$$

Then

$$\varphi(z, w) = \sum_{m=0}^{\infty} w^m P_m(z).$$

Treating $\varphi(z, w)$ as a function of the single variable w, we apply Cauchy's inequalities for the coefficients of a power series; it follows that for any $t > 0$

$$M_G(1; P_m) \leqq M_G(t; f)/t^m. \tag{3.1.5}$$

By the definition of ρ and σ_G it follows that, if $0 < \rho < \infty$ and $\sigma_G < \infty$, for any $\varepsilon > 0$ there exists a constant C_ε such that for all $t > 0$

$$\ln^+ M_G(t; f) \leqq (\sigma_G + \varepsilon) t^\rho + \ln C_\varepsilon.$$

Hence by (3.1.5) it follows that

$$M_G(1; P_m) \leqq C_\varepsilon e^{(\sigma_G+\varepsilon)t^\rho} t^{-m}.$$

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[2] Special cases of this theorem are found in Sire [1] and Temljakov [1].

[3] Recall that $r^k = r_1^{k_1} \ldots r_n^{k_n}$ (see the Introduction, § 1).

Minimizing the right-hand side of this inequality and using (3.1.3), we find that

$$M_G(1; P_m) \leqq C_\varepsilon(e(\sigma_G + \varepsilon)\rho/m)^{m/\rho}. \tag{3.1.6}$$

We now estimate the coefficients of the polynomial $P_m(z)$. For $\|k\| = m$ and any $r \in G$ we have

$$|a_k| \leqq M_{P_m}(r)/r^k.$$

Minimizing the right-hand side of this inequality over all $r \in R_+^n$, we see that for any k

$$|a_k| \leqq M_G(1; P_m)/d_k(G).$$

Together with (3.1.6), this implies that

$$|a_k| \leqq \frac{C_\varepsilon}{d_k(G)}\left(\frac{e\rho(\sigma_G + \varepsilon)}{\|k\|}\right)^{\|k\|/\rho}$$

and

$$(|a_k|d_k(G))^{1/\|k\|}\|k\|^{1/\rho} \leqq C_\varepsilon^{1/\|k\|}\{e\rho(\sigma_G + \varepsilon)\}^{1/\rho}. \tag{3.1.7}$$

All this holds on the assumption that $\sigma_G < \infty$. If we relax this restriction, then, as is readily seen, the above estimates remain meaningful if we replace ρ by an arbitrary number $\rho + \varepsilon$, $\varepsilon > 0$, and $\sigma_G + \varepsilon$ by an arbitrary positive number, say unity. Instead of (3.1.7), we then get the inequality

$$(|a_k|d_k(G))^{1/\|k\|}\|k\|^{1/(\rho+\varepsilon)} \leqq C_\varepsilon^{1/\|k\|}(e(\rho + \varepsilon))^{1/(\rho+\varepsilon)}. \tag{3.1.8}$$

If $\sigma_G < \infty$, it follows from (3.1.7) that

$$\overline{\lim_{\|k\|\to\infty}}\{(|a_k|d_k(G))^{1/\|k\|}\|k\|^{1/\rho}\} \leqq (e\rho(\sigma_G+\varepsilon))^{1/\rho}. \tag{3.1.9}$$

In the general case, it follows from (3.1.8) that

$$\overline{\lim_{\|k\|\to\infty}} \frac{\ln|a_k| + \ln d_k(G)}{\|k\|\ln\|k\|} \leqq -\frac{1}{\rho+\varepsilon},$$

and so

$$\overline{\lim_{\|k\|\to\infty}} \frac{\|k\|\ln\|k\|}{-\ln|a_k| - \ln d_k(G)} \leqq \rho+\varepsilon. \tag{3.1.10}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (3.1.9) and (3.1.10) that

$$\overline{\lim_{\|k\|\to\infty}}\{(|a_k|d_k(G))^{1/\|k\|}\|k\|^{1/\rho}\} \leqq (e\rho\sigma_G)^{1/\rho} \tag{3.1.11}$$

and

$$\overline{\lim_{\|k\|\to\infty}} \frac{\|k\|\ln\|k\|}{-\ln|a_k| - \ln d_k(G)} \leqq \rho. \tag{3.1.12}$$

Now let

$$\overline{\lim_{\|k\|\to\infty}} \{(|a_k|d_k(G))^{1/\|k\|}\|k\|^{1/\rho}\} = \kappa < \infty.$$

We claim that $\kappa \geqq (e\rho\sigma_G)^{1/\rho}$. Indeed, it follows from the definition of κ that for any $\varepsilon > 0$ there exist m_ε such that, for all k with $\|k\| > m_\varepsilon$,

$$(|a_k|d_k(G))^{1/\|k\|}\|k\|^{1/\rho} \leqq \kappa + \varepsilon,$$

and consequently

$$d_k(G)|a_k| \leqq \left(\frac{\kappa+\varepsilon}{\|k\|^{1/\rho}}\right)^{\|k\|}. \tag{3.1.13}$$

Therefore

$$\begin{aligned} M_G(t;f) &\leqq \max_{r\in G} \sum_{\|k\|=0}^{\infty} |a_k| r^k t^{\|k\|} \leqq \sum_{\|k\|=0}^{\infty} t^{\|k\|}|a_k|d_k(G) \\ &\leqq \sum_{\|k\|\geqq m_\varepsilon} t^{\|k\|}\left(\frac{\kappa+\varepsilon}{\|k\|^{1/\rho}}\right)^{\|k\|} + \sum_{\|k\|\leqq m_\varepsilon} t^{\|k\|}|a_k|d_k(G) \\ &\leqq \sum_{m=0}^{\infty} t^m(1+m)^n\left(\frac{(\kappa+\varepsilon)^\rho}{m}\right)^{m/\rho} + c_1 t^{m_\varepsilon} + c_2, \end{aligned} \tag{3.1.14}$$

where c_1 and c_2 are constants. Set $N(t) = (2t(\kappa+\varepsilon))^\rho$. Then it is readily seen that for $m \geqq N(t)$

$$t^m\left(\frac{(\kappa+\varepsilon)^\rho}{m}\right)^{m/\rho} \leqq 2^{-m},$$

and by (3.1.4), for any m, in particular, for $m \leqq N(t)$,

$$t^m\left(\frac{(\kappa+\varepsilon)^\rho}{m}\right)^{m/\rho} \leqq \exp\left\{\frac{(\kappa+\varepsilon)^\rho}{e\rho}t^\rho\right\}.$$

Consequently

$$\begin{aligned} \sum_{m=0}^{\infty} t^m(1+m)^n\left(\frac{(\kappa+\varepsilon)^\rho}{m}\right)^{m/\rho} &= \sum_{m<N(t)} + \sum_{m\geqq N(t)} \\ &\leqq \sum_{m\geqq N(t)} \frac{(1+m)^n}{2^m} + \exp\left\{\frac{(\kappa+\varepsilon)^\rho}{e\rho}t^\rho\right\} \sum_{m\leqq N(t)} (1+m)^n \\ &\leqq \sum_{m\geqq N(t)} \frac{(1+m)^n}{2^m} + (1+N(t))^{n+1}\exp\left\{\frac{(\kappa+\varepsilon)^\rho}{e\rho}t^\rho\right\}. \end{aligned} \tag{3.1.15}$$

Hence by (3.1.14) we conclude that for all $t > 0$ and certain constants c_1, c_2, c_3 and c_4

$$M_G(t;f) \leqq c_1 + c_2 t^{m_\varepsilon} + (c_3 + c_4 t^{\rho(n+1)})\exp\left\{\frac{(\kappa+\varepsilon)^\rho}{e\rho}t^\rho\right\}. \tag{3.1.16}$$

Combining this with (3.1.11), we see that $\kappa = (e\rho\sigma_G)^{1/\rho}$. We have thus proved (3.1.2).

The estimates for the function $M_G(t;f)$ established in the proof of (3.1.6) remain valid when the numbers $\kappa+\varepsilon$ and ρ are replaced by 1 and $\gamma+\varepsilon$, respectively, where

$$\gamma=\varlimsup_{\|k\|\to\infty}\frac{\|k\|\ln\|k\|}{\ln\left(1/|a_k|d_k(G)\right)},$$

ε being an arbitrary positive number. To prove this, it suffices to observe that, by the definition of γ, for any $\varepsilon>0$ there exists m_ε such that for $\|k\|>m_\varepsilon$

$$d_k(G)|a_k|\leqq\|k\|^{-\|k\|/(\gamma+\varepsilon)},$$

and these are precisely inequalities (3.1.13) with $\rho=\gamma+\varepsilon$ and $\kappa+\varepsilon=1$. Thus together with (3.1.16) we have the inequality

$$M_G(t;f)\leqq c_1+c_2t^{m_\varepsilon}+(c_3+c_4t^{(\gamma+\varepsilon)(n+1)})\exp\left\{\frac{t^{\gamma+\varepsilon}}{e(\gamma+\varepsilon)}\right\},$$

whence it follows that $\rho\leqq\gamma+\varepsilon$, and so $\rho\leqq\gamma$. By (3.1.10) this proves that $\rho=\gamma$, i. e.

$$\rho=\varlimsup_{\|k\|\to\infty}\frac{\|k\|\ln\|k\|}{-\ln|a_k|-\ln|d_k(G)|}.$$

We saw in Chapter 2, § 6, that the order ρ is independent of the choice of the domain G. Taking $G=\{r:0<r_i<1,\ i=1,\cdots,n\}$, we see that $d_k(G)=1$ for any k, and hence

$$\rho=\varlimsup_{\|k\|\to\infty}\frac{\|k\|\ln\|k\|}{-\ln|a_k|}.$$

This completes the proof.

In applications of formula (3.1.2) it is useful to remember that, as a direct calculation shows, for $G=\{r:|r|<1\}$,

$$d_k(G)=\sqrt{k_1^{k_1}\cdots k_n^{k_n}/\|k\|^{\|k\|}},$$

and for $G=\{r:\ r_1+\cdots+r_n<1\}$

$$d_k(G)=k_1^{k_1}\cdots k_n^{k_n}/\|k\|^{\|k\|}.$$

2. *Associated orders.* Let $f(z)$ be an entire function of finite order. We define the *hypersurface of associated orders of* $f(z)$ to be the hypersurface $S_\rho(\ln^+M_f)$ of associated orders of the function $\ln^+M_f\in\mathfrak{A}$, and any system of associated orders of the function $\ln^+M_f$ will be called a *system of associated orders of* $f(z)$. Henceforth instead of $S_\rho(\ln^+M_f)$ we shall write $S_\rho(f)$ or simply S_ρ. Similarly, the set $B_\rho(\ln^+M_f)$ figuring in the definition of S_ρ will be denoted by $B_\rho(f)$ or B_ρ. The following theorem indicates the connection between the associated orders and the Taylor coefficients of the function.

THEOREM 3.1.2 (GOL'DBERG [1]). *Positive numbers* $\rho_1,\cdots,\rho_n$ *form a system of associated orders of the entire function* $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k$ *if and only if*

$$\varlimsup_{\|k\|\to\infty} \frac{\frac{k_1}{\rho_1}\ln k_1 + \cdots + \frac{k_n}{\rho_n}\ln k_n}{-\ln|a_k|} = 1. \tag{3.1.17}$$

PROOF.[4] We first prove the following lemma.

LEMMA 3.1.1. *A point* $(b_1,\cdots,b_n)$ *is in the interior* $\mathring{B}_\rho$ *of the set* $B_\rho(f)$ *if and only if*

$$\varlimsup_{\|k\|\to\infty} \frac{\frac{k_1}{b_1}\ln k_1 + \cdots + \frac{k_n}{b_n}\ln k_n}{-\ln|a_k|} < 1.$$

PROOF. Let $(b_1,\cdots,b_n) \in \mathring{B}_\rho$. Then, for some $\varepsilon > 0$ and a constant for C_ε, all $r \in R_+^n$ we have

$$M_f(r) \leqq C_\varepsilon \exp\left\{\sum_{i=1}^{n} r_i^{b_i-\varepsilon}\right\}.$$

Applying Cauchy's inequality, we see that for any $r_1,\cdots, r_n$

$$|a_k| \leqq C_\varepsilon \frac{\exp\{\sum_1^n r_i^{b_i-\varepsilon}\}}{r_1^{k_1}\cdots r_n^{k_n}},$$

and so (see (3.1.3))

$$|a_k| \leqq C_\varepsilon \prod_{i=1}^{n}\left(\frac{e(b_i-\varepsilon)}{k_i}\right)^{k_i/(b_i-\varepsilon)}.$$

It follows from this estimate that

$$\varliminf_{\|k\|\to\infty} \frac{-\ln|a_k|}{\sum_1^n \frac{k_i \ln k_i}{b_i-\varepsilon}} \geqq \varliminf_{\|k\|\to\infty}\left\{\frac{-\ln C_\varepsilon - \sum_1^n \frac{k_i}{b_i-\varepsilon}\ln\frac{e(b_i-\varepsilon)}{k_i}}{\sum_1^n \frac{k_i\ln k_i}{b_i-\varepsilon}}\right\} = 1.$$

Consequently

$$\varlimsup_{\|k\|\to\infty} \frac{\sum_1^n \frac{k_i}{b_i-\varepsilon}\ln k_i}{-\ln|a_k|} \leqq 1$$

and hence

$$\varlimsup_{\|k\|\to\infty} \frac{\sum_1^n \frac{k_i}{b_i}\ln k_i}{-\ln|a_k|} < 1.$$

This proves necessity.

[4] This proof differs only in details from that of the analogous formulas for $n = 1$; see Levin [1].

Now suppose that the numbers $b_1, \cdots, b_n$ satisfy the condition

$$\overline{\lim_{\|k\| \to \infty}} \frac{\sum_1^n \frac{k_i}{b_i} \ln k_i}{-\ln |a_k|} = \kappa < 1.$$

Then for any $\varepsilon > 0$ there exists a number m_ε such that for $\|k\| > m_\varepsilon$

$$-\frac{1}{\ln |a_k|} \sum_{i=1}^n \frac{k_i}{b_i} \ln k_i \leqq \kappa (1 + \varepsilon),$$

and hence

$$|a_k| \leqq \prod_{i=1}^n \left(\frac{1}{k_i}\right)^{k_i/(1+\varepsilon) b_i \kappa}.$$

It follows from these estimates for the coefficients a_k that

$$\begin{aligned} M_f(r) &\leqq \sum_{\|k\| \leqq m_\varepsilon} |a_k| r^k + \sum_{\|k\|=0}^{\infty} \prod_{i=1}^n \left\{ \left(\frac{1}{k_i}\right)^{k_i/(1+\varepsilon) b_i \kappa} r_i^{k_i} \right\} \\ &= \sum_{\|k\| \leqq m_\varepsilon} |a_k| r^k + \prod_{i=1}^n \sum_{k_i=0}^{\infty} r_i^{k_i} \left(\frac{1}{k_i}\right)^{k_i/\kappa b_i (1+\varepsilon)}. \end{aligned}$$

Each sum $\sum_{k_i=0}^{\infty}$ is now estimated, using inequality (3.1.15), as follows:

$$\begin{aligned} \sum_{k_i=0}^{\infty} r_i^{k_i} \left(\frac{1}{k_i}\right)^{k_i/\kappa b_i (1+\varepsilon)} &\leqq (C_1 + C_2 r_i^{2 b_i \kappa (1+\varepsilon)}) \exp \left\{ \frac{r_i^{b_i (1+\varepsilon) \kappa}}{b_i e (1 + \varepsilon) \kappa} \right\} \\ &\leqq C_3 \exp (r_i^{b_i (1+2\varepsilon) \kappa}), \end{aligned}$$

where C_1, C_2 and C_3 are suitable constants. Consequently

$$M_f(r) \leqq \sum_{\|k\| \leqq m_\varepsilon} |a_k| r^k + C_3^n \exp \left\{ \sum_{i=1}^n r_i^{b_i (1+2\varepsilon) \kappa} \right\}.$$

Thus the point b' with coordinates $b_i' = b_i (1 + 3\varepsilon) \kappa$ belongs to the set $B_\rho(f)$ for any $\varepsilon > 0$. Together with each point b', the set B_ρ contains the entire hyperoctant $\{b'' : b_i'' > b_i', i = 1, \cdots, n\}$. Hence, taking the number $\varepsilon > 0$ so small that $(1 + 3\varepsilon) \kappa < 1$, we see that $b \in \mathring{B}_\rho$. Q.E.D.

Now let

$$\overline{\lim_{\|k\| \to \infty}} \frac{\frac{k_1}{\rho_1} \ln k_1 + \cdots + \frac{k_n}{\rho_n} \ln k_n}{-\ln |a_k|} = 1.$$

By the lemma, $(\rho_1, \cdots, \rho_n) \notin \mathring{B}_\rho$, and, as is easily seen, this point is the limit of the points $(\rho_1 t, \cdots, \rho_n t) \in \mathring{B}_\rho$ as $t \to 1 + 0$. Consequently $(\rho_1, \cdots, \rho_n) \in S_\rho$. Conversely, let $(\rho_1, \cdots, \rho_n) \in S_\rho$ and $\rho_1 > 0, \cdots, \rho_n > 0$. Then, by the same lemma,

$$\overline{\lim_{\|k\|\to\infty}} \frac{\frac{k_1}{\rho_1}\ln k_1 + \cdots + \frac{k_n}{\rho_n}\ln k_n}{-\ln|a_k|} \geqq 1, \tag{3.1.18}$$

and, since the set $\bar{B}_\rho$ is octant-like, the points $(\rho_1 t, \cdots, \rho_n t)$ all belong to the domain $\mathring{B}_\rho$ for $t > 1$. Therefore

$$\overline{\lim_{\|k\|\to\infty}} \frac{\frac{k_1}{t\rho_1}\ln k_1 + \cdots + \frac{k_n}{t\rho_n}\ln k_n}{-\ln|a_k|} < 1 \qquad \forall\, t > 1$$

and

$$\overline{\lim_{\|k\|\to\infty}} \frac{\frac{k_1}{\rho_1}\ln k_1 + \cdots + \frac{k_n}{\rho_n}\ln k_n}{-\ln|a_k|} < t \qquad \forall\, t > 1.$$

Consequently

$$\overline{\lim_{\|k\|\to\infty}} \frac{\sum_1^n \frac{k_i}{\rho_i}\ln k_i}{-\ln|a_k|} \leqq 1.$$

Hence by (3.1.18) it follows that

$$\overline{\lim_{\|k\|\to\infty}} \frac{\sum_1^n \frac{k_i}{\rho_i}\ln k_i}{-\ln|a_k|} = 1.$$

REMARK. It is not hard to see that

$$\lim_{\|k\|\to\infty} \frac{\frac{k_1}{\rho_1}\ln k_1 + \cdots + \frac{k_n}{\rho_n}\ln k_n}{(k_1/\rho_1 + \cdots + k_n/\rho_n)\ln(k_1 + \cdots + k_n)} = 1.$$

Thus the equality (3.1.17) in the statement of Theorem 3.1.2 may be replaced by

$$\overline{\lim_{\|k\|\to\infty}} \frac{(k_1/\rho_1 + \cdots + k_n/\rho_n)\ln(k_1 + \cdots + k_n)}{-\ln|a_k|} = 1.$$

A similar modification can be made in the statement of Lemma 3.1.1.

The hypersurface $S_\rho(f)$ of associated orders of an entire function $f(z)$ obviously possesses all the properties established in § 6 of Chapter 2 for hypersurfaces of associated orders of functions in class $\mathfrak{A}$. Moreover, as we shall presently show, these properties are characteristic, even though the functions $\ln^+ M_f(r)$ comprise only a subclass of $\mathfrak{A}$.

THEOREM 3.1.3 (RONKIN [4])[5]. *A hypersurface $S \subset R_+^n$ which is the boundary of*

[5] See also Fuks [1], § 26.

a domain $B \subset R^n_+$ is the hypersurface of associated orders for some entire function $f(z)$ if and only if the image B^{-1} of B under the mapping $a'_i = 1/a_i$, $i = 1, \cdots, n$, is a complete convex domain.

PROOF. Necessity is an obvious consequence of Theorem 2.6.1. To prove sufficiency, assume that the domain B^{-1} is complete and convex, and consider its support function $H_{B^{-1}}(\lambda)$. Since B^{-1} is complete and lies in R^n_+, it follows that the hypersurface $S^{-1} = \partial B^{-1} \backslash \partial R^n_+$ is uniquely determined by the values of the function $H_{B^{-1}}(\lambda)$ for $\lambda \in R^n_+$. (The function $H_{B^{-1}}(\lambda)$ is allowed to take the value $+\infty$.)

Consider the function

$$f(z) = \sum_{\|k\|=2}^{\infty} a_k z^k, \tag{3.1.19}$$

whose coefficients are defined by

$$a_k = \exp\{- H_{B^{-1}}(\lambda^{(k)})|k| \ln \|k\|\},$$

where $\lambda^{(k)} = k/|k|$. It follows from the definition of support functions that for $a' \in B^{-1}$

$$\overline{\lim_{\|k\|\to\infty}} \frac{\langle k, a'\rangle \ln\|k\|}{-\ln|a_k|} = \overline{\lim_{\|k\|\to\infty}} \frac{\langle k, a'\rangle}{H_{B^{-1}}(\lambda^{(k)})|k|} = \overline{\lim_{\|k\|\to\infty}} \frac{\langle \lambda^{(k)}, a'\rangle}{H_{B^{-1}}(\lambda^{(k)})} < 1. \tag{3.1.20}$$

Hence by Lemma 3.1.1, in view of the Remark following Theorem 3.1.2, we conclude that $(1/a'_1, \cdots, 1/a'_n) \in \mathring{B}_\rho(f)$, i.e.

$$B \subset \mathring{B}_\rho(f). \tag{3.1.21}$$

Now let $(1/a'_1, \cdots, 1/a'_n) \in \mathring{B}_\rho(f)$. Then

$$1 > \overline{\lim_{\|k\|\to\infty}} \frac{\langle k, a'\rangle}{-\ln|a_k|} \ln\|k\| = \overline{\lim_{\|k\|\to\infty}} \frac{\langle \lambda^{(k)}, a'\rangle}{H_{B^{-1}}(\lambda^{(k)})}.$$

Conseuently for some $\varepsilon > 0$ and $m_\varepsilon > 0$ and all k such that $\|k\| > m_\varepsilon$ we have

$$\langle \lambda^{(k)}, a'\rangle < (1 - \varepsilon) H_{B^{-1}}(\lambda^{(k)}). \tag{3.1.22}$$

Note that if the domain B^{-1} is unbounded, then, in view of its convexity and completeness, it is the cartesian product of some bounded complete convex domain G of dimension $l < n$ and the interior of the hyperoctant R^{n-l}_+. Thus the set

$$D = \{\lambda : |\lambda| = 1, H_{B^{-1}}(\lambda) < \infty\}$$

always consists of points of the unit sphere lying within the corresponding hyperoctant R^l_+ $(0 \leqq l < n)$, and the restriction of the function $H_{B^{-1}}(\lambda)$ to D coincides with the support function of some bounded complete complex domain $G \subset R^l_+$. The points $\lambda^{(k)}$, $\|k\| > m_\varepsilon$, in the set D obviously constitute a dense subset of D. the function $H_{B^{-1}}(\lambda)$ is continuous on D, and it therefore follows from (3.1.22) that for any $\lambda \in R^n_+$, $|\lambda| = 1$,

$$\langle \lambda, a' \rangle \leqq (1 - \varepsilon) H_{B^{-1}}(\lambda).$$

Consequently $a' \in B^{-1}$ and $(1/a_1', \cdots, 1/a_n') \in B$, and so $\mathring{B}_\rho(f) \subset B$. In view of (3.1.21) this proves that for the function $f(z)$ defined by (3.1.19) we have $\mathring{B}_\rho(f) = B$ and $S_\rho(f) = S$. Q.E.D.

3. *Associated types.* Let $\rho_1, \cdots, \rho_n$ be a system of associated orders of an entire function $f(z)$. We shall say that the function $f(z)$ is of *minimal, normal* or *maximal type for associated orders* $\rho_1, \cdots, \rho_n$ according to whether $B_\sigma(\ln^+ M_f, \rho) = R_+^n$, $B_\sigma(\ln^+ M_f, \rho) \neq R_+^n$ and $B_\sigma(\ln^+ M_f, \rho) \neq \emptyset$, or $B_\sigma(\ln^+ M_f, \rho) = \emptyset$, where $B_\sigma(\ln^+ M_f, \rho)$ is the set figuring in the definition of the hypersurface of associated types of the function $\ln^+ M_f(r) \in \mathfrak{A}$ for orders $\rho_1, \cdots, \rho_n$. The *hypersurface of associated types of the function* $f(z)$ *for associated orders* $\rho_1, \cdots, \rho_n$ is defined to be the hypersurface $S_\sigma(\ln^+ M_f, \rho)$ of associated types of the function $\ln^+ M_f(r) \in \mathfrak{A}$ for orders $\rho_1, \cdots, \rho_n$. Henceforth instead of $S_\sigma(\ln^+ M_f, \rho)$ we shall write $S_\sigma(f)$ or simply S_σ, and instead of $B_\sigma(\ln^+ M_f, \rho)$ we shall write $B_\sigma(f)$ or B_σ. Any system of numbers $\sigma_1, \cdots, \sigma_n$ such that $(\sigma_1, \cdots, \sigma_n) \in S_\sigma(f)$ will be called a *system of associated types of the function* $f(z)$ *for associated orders* $\rho_1, \cdots, \rho_n$.

Assertions analogous to Lemma 3.1.1 and Theorems 3.1.2 and 3.1.3 are valid for associated types.

LEMMA 3.1.2 *A point* $(b_1, \cdots, b_n)$ *is in the interior* B_σ *of the set* $B_\sigma(f)$ *for an entire function* $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k$ *if and only if*

$$\overline{\lim_{\|k\| \to \infty}} \left(|a_k| \prod_{i=1}^{n} \left(\frac{k_i}{e b_i \rho_i} \right)^{k_i/\rho_i} \right)^{1/\|k\|} < 1.$$

THEOREM 3.1.4. *Positive numbers* $\sigma_1, \cdots, \sigma_n$ *are a system of associated types for associated orders* $\rho_1, \cdots, \rho_n$ *of the entire function* $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^k$ *if and only if*

$$\overline{\lim_{\|k\| \to \infty}} \left(|a_k| \prod_{i=1}^{n} \left(\frac{k_i}{\sigma_i e \rho_i} \right)^{k_i/\rho_i} \right)^{1/\|k\|} = 1.$$

The proofs of Lemma 3.1.2 and Theorem 3.1.4 are omitted, since they differ only in details from those of Lemma 3.1.1 and Theorem 3.1.2.

THEOREM 3.1.5 (RONKIN [4]).[6] *A hypersurface* $S \subset R_+^n$ *which is the boundary of a domain* $B \subset R_+^n$ *is the hypersurface of associated types for associated orders* $\rho_1 > 0, \cdots, \rho_n > 0$ *of some entire function* $f(z)$ *if and only if the image* B_* *of the domain* $B \subset R_{(a)}^n$ *under the mapping* $b_i = \ln a_i$, $i = 1, \cdots, n$, *is an octant-like convex domain.*

PROOF. Necessity is an obvious consequence of Theorem 2.6.2. To prove

[6] See also Fuks [1], § 26.

sufficiency, assume that B_* is an octant-like convex domain. In view of the special character of the domain B_*, it is convenient to replace the support function $H_{B_*}(\lambda)$ by the function

$$H^{B_*}(\lambda) = \inf_{b \in B_*} \langle b, \lambda \rangle,$$

which we shall again call the support function.

Let D^{B_*} be the set of points λ at which $H^{B_*}(\lambda) > -\infty$, and consider some sequence of points $\{\lambda^{(j)}\}_1^\infty$ dense in D^{B_*}. Then the domain B_* is the intersection $\bigcap_1^\infty B_j$, where $B_j = \{b: \langle \lambda^{(j)}, b \rangle > H^{B_*}(\lambda^{(j)})\}$. We claim that for each domain B_j there is at least one entire function $f_j(z)$ whose hypersurface of associated types for associated orders $\rho_1, \cdots, \rho_n$ is taken by the mapping $b_j = \ln a_j$, $j = 1, \cdots, n$, into the boundary of the domain B_j, i.e. into the hypersurface

$$S_j = \{b: \langle b, \lambda^{(j)} \rangle = H^{B_*}(\lambda^{(j)})\}.$$

To prove this for fixed j, consider a sequence of indices $k^{(m)} = (k_1^{(m)}, \cdots, k_n^{(m)})$ such that

$$\lim_{m \to \infty} \frac{k_i^{(m)}/\rho_i}{(\sum_1^n (k_i^{(m)}/\rho_i)^2)^{1/2}} = \frac{\lambda_i^{(j)}}{|\lambda^{(j)}|}.$$

Now consider the function

$$f_j(z) = \sum_{m=1}^{\infty} a_m z^{k^{(m)}}, \tag{3.1.23}$$

where

$$a_m = \prod_{i=1}^n \left(\frac{k_i^{(m)}}{e\rho_i}\right)^{-k_i^{(m)}/\rho_i} \exp\left\{H^{B_*}\left(\frac{\lambda^{(j)}}{|\lambda^{(j)}|}\right)\sqrt{\sum_{i=1}^n \left(\frac{k_i^{(m)}}{\rho_i}\right)^2}\right\}.$$

Thanks to the special choice of the sequence $k^{(m)}$, for $b \in S_j$ we have

$$\overline{\lim_{m \to \infty}} \left(a_m \prod_{i=1}^n \left(\frac{k_i^{(m)}}{e^{b_i}\rho_i e}\right)^{k_i^{(m)}/\rho_i}\right)^{1/\|k\|^{(m)}}$$

$$= \lim_{m \to \infty} \left(\exp\left\{-\sum_{i=1}^n \frac{k_i^{(m)}}{\rho_i} b_i + H^{B_*}\left(\frac{\lambda^{(j)}}{|\lambda^{(j)}|}\right)\sqrt{\sum_{i=1}^n \left(\frac{k_i^{(m)}}{\rho_i}\right)^2}\right\}\right)^{1/\|k^{(m)}\|} = 1.$$

Consequently for any $b \in S_j$ the numbers $e^{b_1}, \cdots, e^{b_n}$ form a system of associated types for the function $f_j(z)$.

Now let $\{b^{(m)}\}_1^\infty$ be a sequence of points dense in B_*. Since $B = \bigcap_1^\infty B_j$, it follows that every point $b^{(m)}$ belongs to all domains B_j. Hence for any point $b^{(m)}$ and any function $f_j(z)$ defined over the domain B_j by (3.1.23) there exists a constant $C(j, m) > 0$ such that for all $r_1 \geqq 0, \cdots, r_n \geqq 0$ we have

$$\ln M_{f_j}(r) \leqq \ln C(j, m) + \sum_{i=1}^n e^{b_i^{(m)}} r_i^{\rho_i}. \tag{3.1.24}$$

Set

$$C_j = \frac{1}{2^j}\left\{\max_{1\leqq i,\, l\leqq j} C(i,l)\right\}^{-1} \tag{3.1.25}$$

and consider the function

$$f(z) = \sum_{j=1}^{\infty} C_j f_j(z). \tag{3.1.26}$$

It follows from (3.1.24) — (3.1.26) that, for every point $b^{(m)}$, there is a constant $C(m)$ such that, for all $r \in R_+^n$,

$$\ln M_f(r) \leqq C(m) + \sum_{i=1}^{n} e^{b_i^{(m)}} r_i^{\rho_i}.$$

Consequently $(e^{b_1^{(m)}}, \cdots, e^{b_n^{(m)}}) \in B_\sigma(f)$ Let $B_{\sigma,*}$ denote the image of the domain $B_\sigma(f) \subset R^n_{(a)}$ under the mapping $b_i = \ln a_i$, $i = 1, \cdots, n$. Since the sequence $\{b^{(m)}\}$ is dense in B_*, it follows from $(e^{b_1^{(m)}}, \cdots, e^{b_n^{(m)}}) \in B_\sigma(f)$ that

$$\bar{B}_* \subset \bar{B}_{\sigma,*}. \tag{3.1.27}$$

On the other hand, since the functions $f_j(z)$ have Taylor series with positive coefficients, we have $\bar{B}_{\sigma,*} \subset B_j$, $j = 1,2, \cdots$. Consequently

$$\bar{B}_{\sigma,*} \bigcap_{j=1}^{\infty} \bar{B}_j = \bar{B}_*. \tag{3.1.27'}$$

From (3.1.27) and (3.1.27′) we infer that $\bar{B}_{\sigma,*} = \bar{B}_*$, and consequently $\bar{B}_\sigma(f) = \bar{B}$ and $S_\sigma(f) = S$. Q.E.D.

We end this subsection with a frequently encountered special case of Theorem 3.1.4.

Numbers $\sigma_1 > 0, \cdots, \sigma_n > 0$ are associated types for associated orders of an entire function $f(z) = \sum_{\|k\|=0}^{\infty} a_k z_k/k!$ if and only if

$$\overline{\lim_{\|k\|\to\infty}} \left(\frac{|a_k|}{\sigma_1^{k_1} \cdots \sigma_n^{k_n}}\right)^{1/\|k\|} = 1. \tag{3.1.28}$$

4. Upper order and upper type. We define the *upper order $\bar{\rho}_i = \bar{\rho}_i(f)$ of an entire function $f(z)$ in the variable z_i* to be the order of the function $\ln^+ M_f(r) \in \mathfrak{A}$ in the variable r_i; the *upper type $\bar{\sigma}_i = \bar{\sigma}_i(f, \bar{\rho}_i)$ of $f(z)$ for order $\bar{\rho}_i$ in the variable z_i* is defined as the upper type of $\ln^+ M_f(r)$ in the variable r_i for order $\bar{\rho}_i$. The type of $\ln^+ M_f(r)$ for order $\bar{\rho}_i$ in r_i is denoted here by $\sigma_i(r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n; f, \bar{\rho}_i)$. The results of Chapter 2, § 6, for types and orders in one variable of functions of class $\mathfrak{A}$ are of course valid in this special case. In particular, we have the following relations:

$$\rho_f \leqq \bar{\rho}_1(f) + \cdots + \bar{\rho}_n(f), \tag{3.1.29}$$

$$\bar{\rho}_i(f) = \inf_{(\rho_1, \cdots, \rho_n) \in S\rho(f)} \rho_i. \tag{3.1.30}$$

However, the analogs of (3.1.29) and (3.1.30) for types $\bar{\sigma}_i$ are false; this will be shown by the following examples of entire functions, which will also furnish a proof of the corresponding remark in Chapter 2, § 6.

EXAMPLE 1. Let $f(w)$ be an entire function of $w \in C^1$, of order $\rho \in (1/2, 1)$. Consider an arbitrary rational number $m/n \in (\rho,1)$. It follows from Cauchy's inequality that

$$|z_1 z_2|^{m/n} \leqq \frac{m|z_1| + |z_2|^m + n - m - 1}{n}.$$

Consequently for any z_1 and z_2 the entire function $\varphi(z_1, z_2) = f(z_1 z_2)$ satisfies the following estimate:

$$\begin{aligned} |\varphi(z_1, z_2)| &= |f(z_1 z_2)| \leqq \text{const} \cdot \exp\left(|z_1 z_2|^{m/n}\right) \\ &\leqq \text{const} \cdot \exp(m|z_1|/n) \cdot \exp((|z_2|^m + n - m - 1)/n), \end{aligned}$$

whence it follows that $\bar{\sigma}_1(\varphi; 1) \leqq m/n$. Similarly, $\bar{\sigma}_2(\varphi; 1) \leqq m/n$. On the other hand, it is clear that the order of the function $\varphi(z_1, z_2)$ is $\rho_\varphi = 2\rho > 1$, and thus it cannot have finite G-type for order 1.

EXAMPLE 2. Consider the function

$$f(z_1, z_2) = \sum_{m=2}^{\infty} \frac{z_1^m [\sqrt{\ln m}] z_2^m}{(m[\sqrt{\ln m}])!\, m!}.$$

Since

$$\overline{\lim_{m\to\infty}} \frac{m[\sqrt{\ln m}] \ln (m[\sqrt{\ln m}]) + m \ln m}{\ln ((m[\sqrt{\ln m}])!\, m!)} = 1,$$

the numbers 1, 1 are associated orders of the function $f(z_1, z_2)$. Let us determine the hypersurface of associated types for orders 1,1. We have

$$\overline{\lim_{m\to\infty}} (\sigma_1^{-m[(\ln m)^{1/2}]} \sigma_2^{-m})^{1/(m+m[(\ln m)^{1/2}])} = \frac{1}{\sigma_1}.$$

Consequently (see (3.1.28)) the curve of associated types S_σ is in this case the boundary of the quadrant $\{a_1 > 1,\ a_2 > 0\}$, and therefore

$$\inf_{(\sigma_1, \sigma_2) \in S_\sigma} \sigma_1 = 1.$$

On the other hand,

$$\begin{aligned} \bar{\sigma}_1 &= \sup_{0<r_2<\infty} \sigma_1(r_2; f, 1) = \sup_{0<r_2<\infty} \overline{\lim_{m\to\infty}} (r_2^m/m!)^{1/m[(\ln m)^{1/2}]} \\ &= \sup_{0<r_2<\infty} \lim_{m\to\infty} r_2^{1/[\sqrt{\ln m}]} \cdot \exp\left((m - m \ln m)/m[(\ln m)^{1/2}]\right) = 0. \end{aligned}$$

Thus our function satisfies the inequality

$$\bar{\sigma}_1 < \inf_{(\sigma_1, \sigma_2) \in S_\sigma} \sigma_1.$$

The upper order $\bar{\rho}_i$ of an entire function has a very simple expression in terms of the Taylor coefficients a_k (see Maergoĭz [3]):

$$\bar{\rho}_i = \overline{\lim_{\|k\|\to\infty}} \frac{k_i \ln k_i}{-\ln|a_k|}. \tag{3.1.31}$$

To prove this formula, we first observe that, since $f(z)$ is an entire function, all the coefficients a_k, with the possible exception of a finite number, satisfy the inequality $|a_k| < 1$. Therefore for any $\rho_1 > 0, \cdots, \rho_n > 0$ we have

$$\sup_i \overline{\lim_{\|k\|\to\infty}} \frac{k_i \ln k_i}{-\rho_i \ln|a_k|} \leqq \overline{\lim_{\|k\|\to\infty}} \frac{\sum_1^n \frac{k_i}{\rho_i} \ln k_i}{-\ln|a_k|} \leqq \sum_{i=1}^n \overline{\lim_{\|k\|\to\infty}} \frac{\frac{k_i}{\rho_i} \ln k_i}{-\ln|a_k|}. \tag{3.1.32}$$

Now if $(\rho_1, \cdots, \rho_n) \in S_\rho(f)$, then it follows from (3.1.32) and Theorem 3.1.2 that

$$\frac{1}{\rho_i} \overline{\lim_{\|k\|\to\infty}} \frac{k_i \ln k_i}{-\ln|a_k|} \leqq 1 \leqq \sum_{j=1}^n \frac{1}{\rho_j} \overline{\lim_{\|k\|\to\infty}} \frac{k_j \ln k_j}{-\ln|a_k|}.$$

Consequently

$$\overline{\lim_{\|k\|\to\infty}} \frac{k_i \ln k_i}{-\ln|a_k|} \leqq \rho_i \leqq \overline{\lim_{\|k\|\to\infty}} \frac{k_i \ln k_i}{-\ln|a_k|} + \sum_{j\neq i} \frac{\rho_i}{\rho_j} \lim_{\|k\|\to\infty} \frac{k_j \ln k_j}{-\ln|a_k|}$$

and thus

$$\begin{aligned} \overline{\lim_{k_i\to\infty}} \frac{k_i \ln k_i}{-\ln|a_k|} &\leqq \inf_{(\rho_1,\cdots,\rho_n)\in S_\rho} \rho_i \\ &\leqq \overline{\lim_{\|k\|\to\infty}} \frac{k_i \ln k_i}{-\ln|a_k|} + \inf_{(\rho_1,\cdots,\rho_n)\in S_\rho} \sum_{j\neq i} \frac{\rho_i}{\rho_j} \cdot \sum_{j=1}^n \overline{\lim_{\|k\|\to\infty}} \frac{k_j \ln k_j}{-\ln|a_k|}. \end{aligned} \tag{3.1.33}$$

Since S_ρ is the boundary of an octant-like domain, it follows that for any i

$$\inf_{(\rho_1,\cdots\rho_n)\in S_\rho} \sum_{j\neq i} \frac{\rho_i}{\rho_j} = 0,$$

and it follows from (3.1.33) that

$$\inf_{(\rho_1,\cdots\rho_n)\in S_\rho(f)} \rho_i = \lim_{\|k\|\to\infty} \frac{k_i \ln k_i}{-\ln|a_k|}.$$

To prove (3.1.31), it is now sufficient to refer to (3.1.30).

§ 2. Growth of an entire function in one of the variables, the others being fixed

1. *Order and type.* Let $f(z,w)$, $z \in C^n$, $w \in C^1$, be an entire function. We denote

$$M_f(r_1, \cdots, r_{i-1}, z_i, \cdots, z_n, t) = \max_{\substack{|w|=t,\,|z_j|=r_j,\\ j=1,\cdots,i-1}} |f(z, w)|.$$

We define the *order of the function* $f(z, w)$ *in the variable* w to be

$$\rho_f(z) = \overline{\lim_{t\to\infty}} \frac{\ln \ln^+ M_f(z, t)}{\ln t},$$

i.e. the ordinary order of $\ln^+ M_f(z,t)$ as a function of t. Similarly, the *type of the function* $f(z, w)$ *in the variable* t *for upper order* $\bar{\rho}_w$ [7] is

$$\sigma_f(z) = \sigma_f(z; \bar{\rho}_w) = \overline{\lim_{t\to\infty}} \frac{\ln^+ M_f(z, t)}{t^{\bar{\rho}_w}},$$

i.e. the corresponding type of the function $\ln^+ M_f(z, t)$. As usual, $\sigma_f^*(z)$ will denote the regularization of the function $\sigma_f(z)$.

We mentioned in § 7 of Chapter 2 that the function $\ln^+ M_f(z, t)$ is in class $\mathfrak{B}$. Thus the order $\rho_f(z)$ and type $\sigma_f(z)$ satisfy the assertions of Theorems 2.7.1 through 2.7.4, which are valid for arbitrary functions in $\mathfrak{B}$. Thus the following theorems follow immediately.

THEOREM 3.2.1. [8] *Let* $f(z,w)$, $z \in C^n$, $w \in C^1$, *be an entire function such that* $\rho_f(z) < \infty$ *on some set* E *of positive* Γ*-capacity. Then the upper order* $\bar{\rho}_w(f)$ *of this function in the variable* w *is finite, and* $\rho_f(z) = \bar{\rho}_w(f)$ *throughout* C^n, *except perhaps for a set* $N_\rho(f)$ *of class* F_σ *which must satisfy the following condition*:

(A) $N_\rho(f) \neq C^n$, *and the intersection of* $N_\rho(f)$ *with any analytic plane* $\{z: z_i = a_i w + b_i, i = 1, \cdots, n\}$, *if not entirely in* $N_\rho(f)$, *has zero capacity.*

THEOREM 3.2.2. *Let* $f(z,w)$ *be an entire function,* $\bar{\rho}_w(f) < \infty$, *and assume that for some* $r_1 > 0, \cdots, r_n > 0$

$$\sigma_w(r_1, \cdots, r_n; \ln^+ M_f, \bar{\rho}_w) < \infty. \tag{3.2.1}$$

Then the function $\ln \sigma_f^*(z)$ *is plurisubharmonic*[9] *in the polydisk* $\{z: |z_i| < r_i, i = 1, \cdots, n\}$, *where for* $R_i < r_i$, $i = 1, \cdots, n$,

$$\sigma_w(R_1, \cdots, R_n; \ln^+ M_f, \bar{\rho}_w) = \sup_{\substack{|z_i|<R_i \\ i=1,\cdots,n}} \sigma_f(z),$$

[7] Here and below, instead of the index of the variable in which the growth is being considered, we shall write the variable itself.

[8] This theorem was first proved, for $z \in C^1$, by Lelong [1]. The general case was considered in Ronkin [5] and [6].

[9] The plurisubharmonicity of $\ln \sigma_f^*(z)$ is not a direct consequence of Theorem 2.7.2, which implies only that $\sigma_f^*(z)$ is plurisubharmonic. Nevertheless it is sufficient to express the function $f(z,w)$ as

$$f(z,w) = \sum_{m=1}^{\infty} g_m(z) w^m$$

and to use the properties of the upper limit of a family of plurisubharmonic functions (p. 95), taking account of the fact that

$$\ln \sigma_f(z) = \overline{\lim_{m\to\infty}} \left\{ \ln m + \frac{\bar{\rho}_w}{m} \ln |g_m(z)| \right\} - \ln \bar{\rho}_w - 1,$$

and that by (3.2.1) the family of plurisubharmonic functions

$$\ln m + (\bar{\rho}_w/m) \ln |g_m(z)|$$

is uniformly bounded on every compact set $k \subset \{z: |z_i| < r_i, i = 1, \cdots, n\}$.

and the set $N_\sigma(f)$ of points z at which $\sigma_f(z) < \sigma_f^*(z)$ is an F_σ set of zero Γ-capacity.

COROLLARY. *Let $f(z,w)$ be an entire function such that $\bar\rho_w < \infty$ and $\bar\sigma_w < \infty$. Then the set of points z at which $\sigma_f(z) < \bar\sigma_w$ is an F_σ set of zero Γ-capacity.*

THEOREM 3.2.3. *Let the entire function $f(z,w)$ have finite order ρ_f. Assume moreover that, on some set $E \subset C^n$ of positive Γ-capacity, $\sigma_f(z) \leqq \gamma$ for some $\gamma < \infty$. Then for any $R > 0$ and some constant C_f*

$$\sigma_w(R, \cdots, R; \ln^+ M_f, \bar\rho_w) \leqq \gamma\, C_f R^{\rho_f - \bar\rho_w}. \tag{3.2.2}$$

Note that if the assumption of Theorem 3.2.3 that the order ρ_f is finite is replaced by the weaker condition that its lower order (in all variables jointly)

$$\rho_f = \varliminf_{R\to\infty} \frac{\ln \ln^+ M_f(R, \cdots, R)}{\ln R}$$

be finite, then inequality (3.2.2) fails; it may even happen that for some $R = R_1 > 0$

$$\sigma_w(R_1, \cdots, R_1; \ln^+ M_f, \bar\rho_w) = 0,$$

while for some other $R = R_2$

$$\sigma_w(R_2, \cdots, R_2; \ln^+ M_f, \bar\rho_w) = \infty.$$

To construct an example of this situation, we first prove

LEMMA 3.2.1.[10] *Given an arbitrary power series $\sum_1^\infty c_m w^m$ ($w \in C^1$) with infinite radius of convergence, there is a sequence of indices m_j, $m_j \to \infty$ as $j \to \infty$, such that the sum of the series $\sum_j c_{m_j} w^{m_j}$ is an entire function of zero lower order.*

PROOF. Set $s_j(t) = \sum_1^j |c_m| t^m$, where t_j is the largest root of the equation $e + s_j(t) = \exp\{t^{1/j}\}$ and $l(j)$ the first natural number for which

$$\sum_{m=l(j)}^{\infty} |c_m| t_j^m < 1.$$

Now set $l_0 = 1$ and $l_m = l(l_{m-1})$, $m = 1,2, \cdots$. Then the required series is

$$\varphi(w) = \sum_{m=0}^{\infty} c_{l_m} w^{l_m}.$$

Indeed, it is readily seen that $t_{l_m} \to \infty$ as $m \to \infty$, and

$$M_\varphi(t_{l_i}) \leqq \sum_{m=0}^{\infty} |c_{l_m}| t_{l_i}^{l_m} = \sum_{m=0}^{i} |c_{l_m}| t_{l_i}^{l_m} + \sum_{m=i+1}^{\infty} |c_{l_m}| t_{l_i}^{l_m}$$

$$\leqq \sum_{m=0}^{l_i} |c_m| t_{l_i}^m + \sum_{m=l_{i+1}} |c_m| t_{l_i}^m \leqq s_{l_i}(t_{l_i}) + 1 = 1 + \exp\{t_{l_i}^{1/l_i}\} - e.$$

[10] This lemma was communicated to us by A. A. Gol'dberg.

Consequently

$$\varliminf_{t\to\infty} \frac{\ln \ln^+ M_\varphi(t)}{\ln t} = \varliminf_{m\to\infty} \frac{\ln \ln^+ M_\varphi(t_{l_m})}{\ln t_{l_m}} = 0.$$

This proves the lemma.

Now apply Lemma 3.2.1 to the series

$$\sum_{m=0}^{\infty} w^{m+m[(\ln m)^{1/2}]}/m!.$$

We get a sequence of indices l_m such that the function

$$\varphi(w) = \sum_{m=0}^{\infty} \frac{1}{(l_m)!} w^{l_m+l_m[(\ln l_m)^{1/2}]}$$

has zero lower order. Then the function

$$\varphi(z, w) = \sum_{m=0}^{\infty} \frac{z^{l_m[(\ln l_m)^{1/2}]} w^{l_m}}{(l_m)!}$$

has zero lower order in both its variables $z \in C^1$, $w \in C^1$ jointly. Now observe that $\bar{\rho}_w = 1$. Indeed,

$$\varliminf_{m\to\infty} \frac{l_m \ln l_m}{\ln (l_m!) - l_m [\sqrt{\ln l_m}] \ln|z|} = 1.$$

We now calculate the type of the function $\varphi(z, w)$ in the variable w for order 1. The result is

$$\sigma_\varphi(z) = \varlimsup_{m\to\infty} (|z|^{l_m[(\ln l_m)^{1/2}]})^{1/l_m} = \begin{cases} \infty & \text{for } |z| > 1, \\ 1 & \text{for } |z| = 1, \\ 0 & \text{for } |z| < 1. \end{cases}$$

Thus the function $\varphi(z, w)$ is the promised example.

Hitherto we have investigated the growth of an entire function $f(z,w)$ in complex one-dimensional analytic planes parallel to the coordinate plane $\{(z, w)\colon z = 0, w \in C^1\}$. Investigation of the growth of entire functions on analytic planes through the origin can also be reduced to this case. Indeed, each such plane in the space C^n may be presented as $\{z\colon z_i = \lambda_i w, i = 1, \cdots, n\}$, where one of the numbers λ_i is unity. Thus investigation of the growth of an entire function $f(z)$ in this plane reduces to the analogous problem for the function $\varphi(\lambda, w) = f(\lambda w)$ in the variable w for fixed $\lambda \in C^n$.

THEOREM 3.2.4. *Let $f(z)$ be an entire function, and let $\varphi(z, w)$, $z \in C^n$, $w \in C^1$, be defined by $\varphi(z, w) = f(wz)$. Assume moreover that the set $E \subset C^{n-1}$ has positive Γ-capacity and $\rho_\varphi(z) < \infty$ for all $z \in \{z:(z_1, \cdots, z_{n-1}) \in E, z_n = 1\}$. Then the following assertions are true.*

(a) *The function $f(z)$ has finite order ρ_f.*

(b) *Throughout* C^{n-1}, *with the possible exception of an* F_σ *set satisfying condition* (A) *of Theorem* 3.2.1, $\rho_\varphi(z_1, \cdots, z_{n-1}, 1) = \rho_f$.

(c) *For all* $(z_1, \cdots, z_{n-1})$ *with the possible exception of some set of zero* Γ*-capacity, the function* $f(wz_1, \cdots, wz_{n-1}, w)$ *has the same type* (*minimal, normal or maximal*) *in the variable* w *for order* ρ_f.

PROOF. It follows from Theorem 3.2.1, applied to the function $\varphi(z_1, \cdots, z_{n-1}, 1, w)$, that the upper order $\bar{\rho}$ of $\varphi(z_1, \cdots, z_{n-1}, 1, w)$ in the variable w is finite. Consequently there exist a constant ρ' and a function $d(z_1, \cdots, z_{n-1})$, bounded in every bounded region of C^{n-1}, such that for all $z_1, \cdots, z_{n-1}$ and w

$$|\varphi(z_1, \cdots, z_{n-1}, 1, w)| = |f(wz_1, \cdots, wz_{n-1}, w)| \leqq d(z_1, \cdots, z_{n-1})\, e^{|w|^{\rho'}}.$$

Taking $|z_1| \leqq 1, \cdots, |z_{n-1}| \leqq 1$ in this inequality and setting

$$\max_{\substack{|z_i| \leqq 1 \\ i=1,\cdots,n}} d(z_1, \cdots, z_{n-1}) = D,$$

we see that $M_f(R, \cdots, R) \leqq D \exp R^{\rho'}$ for all R, and thus the function $f(z)$ has finite order $\rho_f \leqq \rho'$. Since ρ' may be any number greater than $\bar{\rho}$, it follows from the inequality $\rho_f \leqq \rho'$ that $\rho_f \leqq \bar{\rho}$. The converse inequality is obvious. Hence $\rho_f = \bar{\rho}$, and thus part (b) of our theorem is an immediate consequence of Theorem 3.1.1.

Now note that

$$M_\varphi(t, t, \cdots, t, 1, t) = M_f(t^2, \cdots, t^2, t) \leqq M_f(t^2, \cdots, t^2),$$

and consequently the order of the function $\varphi(z_1, \cdots, z_{n-1}, 1, w)$ (in all its variables) is finite. Theorem 3.1.3 is therefore applicable to this function. It follows that if

$$\sigma_\varphi(z_1, \cdots, z_{n-1}, 1; \bar{\rho}) = 0 \tag{3.2.3}$$

on some set of positive Γ-capacity, then this equality holds everywhere in C^{n-1}. Further, assume that on some set of positive Γ-capacity

$$\sigma_\varphi(z_1, \cdots, z_{n-1}, 1; \bar{\rho}) < \infty.$$

Then the set E of all points $(z_1, \cdots, z_{n-1})$ at which this inequality holds also has positive Γ-capacity. This set may obviously be expressed as $E = \bigcup_1^\infty E_m$, where

$$\begin{aligned} E_m &= \{(z_1, \cdots, z_{n-1}, 1): \sigma_\varphi(z_1, \cdots, z_{n-1}, 1) \leqq m\} \\ &= \left\{(z_1, \cdots, z_{n-1}, 1): \lim_{|w| \to \infty} \frac{\ln^+ |f(wz_1, \cdots, wz_{n-1}, w)|}{|w|^{\bar{\rho}}} \leqq m\right\}. \end{aligned}$$

Each E_m is an F_σ set, and therefore (see Chapter 2, § 3) the Γ-capacity of E can be positive only if the Γ-capacity of at least one set E_m is positive. Again applying Theorem 3.2.3, we see that throughout C^{n-1}

$$\sigma_\varphi(z_1, \cdots, z_{n-1}, 1) < \infty,$$

i.e. for all $z_1, \cdots, z_{n-1}$ the type of the function $f(wz_1, \cdots, wz_{n-1}, w)$ in the variable w is at most normal. Together with the preceding remark concerning equality (3.2.3), this immediately implies part (c) of the theorem. Q.E.D.

2. *Indicator.* Under certain restrictions, the assertion that the growth of an entire function $f(z,w)$ in the variable w is the same for almost all z is also valid when the growth is characterized by the *indicator*

$$h_f(\theta; z) = \overline{\lim_{t\to\infty}} \frac{1}{t^{\bar\rho_w}} \ln\left|f(z, te^{i\theta})\right|. \tag{3.2.4}$$

THEOREM 3.2.5. *Let $f(z, w)$ be an entire function, $z \in C^n$, $w \in C^1$, of finite upper order $\bar\rho_w$ in the variable w and finite upper type $\bar\sigma_w$ for this order. Let $h_f(\theta; z)$ be the indicator of $f(z, w)$ in the variable w, and $\bar h_f(\theta)$ its upper indicator in w, defined by*

$$\bar h_f(\theta) = \sup_{z\in C^n} h_f(\theta; z). \tag{3.2.5}$$

Then the set $N_h(f)$ of points z such that, for at least one value of θ,

$$h_f(\theta; z) < \bar h_f(\theta), \tag{3.2.6}$$

is an F_σ set and satisfies condition (A) *of Theorem* 3.2.1.

PROOF. We first prove that the set $N_h^{(\theta)}$ of points z such that inequality (3.2.6) holds for a given θ satisfies condition (A). To this end, we observe that (3.2.4) defines the indicator $h_f(\theta; z)$ as the upper limit of a family of plurisubharmonic functions. It is readily seen that this family is uniformly bounded in every bounded region of the space C^n. Consequently the regularization $h_f^*(\theta; z)$ of the indicator $h_f(\theta; z)$ is a plurisubharmonic function in C^n. From the obvious inequality $h_f(\theta; z) \leqq \bar\sigma_w$ we see that $h_f^*(\theta; z) \leqq \bar\sigma_w$, whence by Theorem 2.1.4 it follows that $h_f^*(\theta; z) \equiv \text{const}$; and, in fact, $h_f^*(\theta; z) \equiv \bar h_f(\theta)$. It follows that the set $N_h^{(\theta)}$ is precisely the set $S_{\mathfrak{F}}$ (see Chapter 2, § 3) for the family of functions

$$\mathfrak{F} = \{t^{-\bar\rho_w} \ln\left|f(z, te^{i\theta})\right|\}_{t>0}.$$

Hence by Theorem 2.3.3 we conclude that $N_h^{(\theta)} \in F_\sigma$ and

$$\Gamma\text{-Cap}\,(N_h^{(\theta)}) = 0. \tag{3.2.7}$$

For $n = 1$, equality (3.2.7) means that the set $N_h^{(\theta)}$ satisfies condition (A).

Now let $n > 1$. Consider the function

$$\varphi_{a,b}(\lambda, w) = f(a_1 + b_1\lambda, \cdots, a_n + b_n\lambda, w) \qquad (\lambda \in C^1)$$

and its upper indicator $\bar h_\varphi(\theta)$. If $\bar h_f(\theta) = \bar h_\varphi(\theta)$, then the set $N_h^{(\theta)}(\varphi)$ obviously coincides with the set of points λ defining the intersection of the set $N_h^{(\theta)}(f)$ with the plane $\{z: z_i = a_i + b_i\lambda, i = 1, \cdots, n\}$. But if $\bar h_\varphi(\theta) < \bar h_f(\theta)$, then we have the obvious inclusion relation

$$\{z: z_i = a_i + \lambda b_i, i = 1, \cdots, n\} \subset N_h^{(\theta)}(f).$$

It is also obvious that $N_h^{(\theta)}(f) \neq C^n$. We have thus proved that the set $N_h^{(\theta)}(f)$ satisfies condition (A) of Theorem 3.2.1.

As the indicator of an entire function of one complex variable w,[11] the indicator $h_f(\theta; z)$ is a trigonometrically convex function (see § 3 of the Introduction) of θ, i. e.

$$h_f(\theta_2; z) \leqq \frac{h_f(\theta_1; z) \sin \bar{\rho}_w(\theta_3 - \theta_2) + h_f(\theta_3; z) \sin \bar{\rho}_w(\theta_2 - \theta_1)}{\sin \bar{\rho}_w(\theta_3 - \theta_1)}$$

for any $\theta_1 < \theta_2 < \theta_3$ and $\theta_3 - \theta_1 \leqq \pi/\bar{\rho}_w$. This inequality clearly remains valid if we replace $h_f(\theta; z)$ by $\bar{h}_f(\theta)$, and so $\bar{h}_f(\theta)$ is also trigonometrically convex. Like convex functions, trigonometrically convex functions are continuous.[12] Consequently both the upper indicator $\bar{h}_f(\theta)$ and the indicators $h_f(\theta; z)$ are continuous functions of θ. Consider some sequence $\{\theta_m\}_1^\infty$ dense in the interval $(0, 2\pi)$, and form the set $N_h^{(\theta_m)}$. By the continuity of the functions $\bar{h}_f(\theta)$ and $h_f(\theta; z)$ in θ, the fact that inequality (3.2.6) holds for some z at a certain point $\theta = \theta_0$ implies that it holds for the same z at one of the points θ_m. Consequently

$$N_h(f) = \bigcup_{m=1}^{\infty} N_f^{(\theta_m)},$$

whence in view of the properties of the set $N_h^{(\theta_m)}$ we conclude that the set $N_h(f)$ satisfies condition (A) and is an F_σ set. This completes the proof.

3. *Construction of an entire function with prescribed exceptional set.* The characterization of the sets $N_\rho(f)$ and $N_\sigma(f)$ provided by Theorems 3.2.1 and 3.2.2 is almost complete for entire functions of two complex variables. This is indicated by the following theorems.

THEOREM 3.2.6 [13] (STAVSKIĬ [3]). *Let $E \subset C^1_{(z)}$ be an F_σ set of zero capacity. Then there exists an entire function $f(z, w)$, $z \in C^1$, $w \in C^1$, of given finite upper order $\bar{\rho}_w = \rho > 0$, such that $E \subset N_\rho(f) \subset \bar{E}$.*

PROOF. Since E is an F_σ set of zero capacity, it may be expressed as a union $E = \bigcup_1^\infty E_l$, where each set E_l is closed, has zero capacity, and $E_l \subset E_{l+1}$ for any l. Without loss of generality we may also assume that $E_l \subset \{z: |z| < e^{l^{1/2}}\}$. For each set E_l, construct a sequence of Čebyšev polynomials $t_m^{(l)}(z)$, i.e. a sequence of polynomials

$$t_m^{(l)}(z) = \prod_{j=1}^{m} (z - z_{m,l}^{(j)}) \qquad (z_{m,l}^{(j)} \in E_l)$$

[11] Concerning indicators of entire functions of one variable, see, for example, Levin [1].

[12] See, for example, Levin [1].

[13] Theorem 3.2.6, whose proof resembles that of Theorem 3.2.7 (Ronkin [1]), was actually proved later than Theorem 3.2.7. We have broken the chronological sequence of these theorems since in this book theorems on orders always precede the corresponding theorems on types.

with least deviation from zero on the set E_l.[14] Denote

$$M_m^{(l)} = \max_{z \in E_l} |t_m^{(l)}(z)|.$$

Since the capacity of any set E_l is zero, it follows from standard properties of the Čebyšev polynomials that for any l

$$\lim_{m \to \infty} (M_m^{(l)})^{1/m} = 0. \tag{3.2.8}$$

Let $\{m_l\}$ be a sequence such that for any l

$$(M_{m_l}^{(l)})^{1/m_l} < l^{-1}.$$

This is possible by (3.2.8). Let $\alpha_l(m)$ denote a solution of the equation

$$\ln (m/\alpha) = \alpha l.$$

This equation is easily seen to have a unique solution. It is also readily shown (see Figure 3) that $\ln(m/\alpha_l(m)) \to +\infty$ as m and l increase. We can therefore extract from the sequence m_l a subsequence $m_i^* = m_{l_i}$ such that for any i

$$\left[\frac{m_i^* \rho}{\alpha_{l_i}(m_i^*)}\right] < \left[\frac{m_{i+1}^* \rho}{\alpha_{l_{i+1}}(m_{i+1}^*)}\right].$$

We claim that the function

$$f(z, w) = \sum_{i=1}^{\infty} \frac{T_i(z) w^{[\beta_i]}}{e^{l_i m_i^*}},$$

where

$$\beta_i = \frac{\rho m_i^*}{\alpha_{l_i}(m_i^*)} = \rho \exp \{l_i \alpha_{l_i}(m_i^*)\}$$

and $T_i(z) = t_{m_i^*}^{(l_i)}(z)$, meets our requirements. Indeed, the upper order of this function in w satisfies the estimate

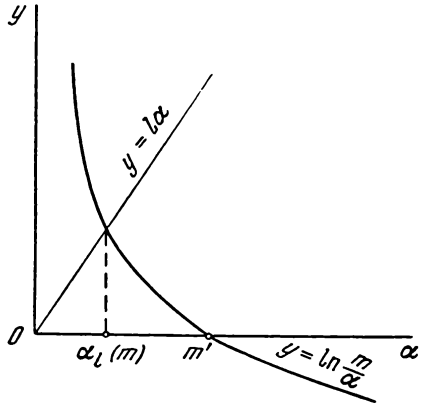

FIGURE 3

$$\bar{\rho}_w = \sup_{|z|<1} \overline{\lim_{i \to \infty}} \frac{[\beta_i] \ln [\beta_i]}{l_i m_i^* - \ln |T_i(z)|} \leqq \overline{\lim_{i \to \infty}} \frac{\frac{\rho m_i^*}{\alpha_{l_i}(m_i^*)} \{\ln \rho + l_i \alpha_{l_i}(m_i^*)\}}{l_i m_i^* - (1 + \sqrt{l_i}) m_i^*} = \rho.$$

On the other hand, by (3.1.31)

$$\bar{\rho}_w \geqq \overline{\lim_{i \to \infty}} \frac{[\beta_i] \ln [\beta_i]}{l_i m_i^*} = \rho.$$

Consequently $\bar{\rho}_w = \rho$. We now determine $\rho_f(z)$ for $z \in E$. We have

[14] Details on Čebyšev polynomials may be found, e.g., in Goluzin [1] or Smirnov and Lebedev [1].

$$\rho_f(z) = \overline{\lim_{i\to\infty}} \frac{[\beta_i]\ln[\beta_i]}{l_i m_i^* - \ln|T_i(z)|} \leqq \overline{\lim_{i\to\infty}} \frac{\rho m_i^*(\ln\rho + l_i\alpha_{l_i}(m_i^*))}{\alpha_{l_i}(m_i^*)\left(l_i m_i^* - \ln M_{m_i^*}^{(l_i)}\right)}$$

$$\leqq \overline{\lim_{i\to\infty}} \frac{\rho(\ln\rho + l_i\alpha_{l_i}(m_i^*))}{\alpha_{l_i}(m_i^*)(l_i + l_i\ln l_i)} = 0.$$

Consequently $\rho_f(z) = 0$ for $z \in E$, and so $E \subset N_\rho(f)$. But if $z \notin \bar{E}$, then

$$\min_{z'\in\bar{E}} |z - z'| = \delta > 0$$

and so

$$\rho_f(z) = \overline{\lim_{i\to\infty}} \frac{[\beta_i]\ln[\beta_i]}{l_i m_i^* - \ln|T_i(z)|} \geqq \overline{\lim_{i\to\infty}} \frac{[\beta_i]\ln[\beta_i]}{l_i m_i^* - m_i^*\ln\delta} = \overline{\lim_{i\to\infty}} \frac{\beta_i\ln\beta_i}{l_i m_i^*} = \rho.$$

Thus $\rho_f(z) = \rho$ for $z \notin \bar{E}$, and this shows that $N_\rho(f) \subset \bar{E}$. Q.E.D.

THEOREM 3.2.7 (RONKIN [1]). *Let $\bar{E}$ be an F_σ set of zero capacity in the plane. Then there exists an entire function $f(z, w)$, $z \in C^1$, $w \in C^1$, with upper order $\bar{\rho}_w$ equal to a given positive number ρ, upper type $\bar{\sigma}_w = 1$, and exceptional set $N_\sigma(f) = \{z: \sigma_f(z) < 1\}$ such that $E \subset N_\sigma(f) \subset \bar{E}$.*

PROOF. We first consider the case in which $\bar{E}$ is not the entire plane, i.e. there exist a point $z^0 \notin \bar{E}$. Since E is an F_σ set of zero capacity, it follows, as in the preceding theorem, that $E = \bigcup_1^\infty E_l$, where each E_l is a closed set of zero capacity, $E_l \subset E_{l+1}$ and $E_l \subset \{z: |z| < e^l\}$ for every l. As before, we construct a sequence of Čebyšev polynomials for each set E_l:

$$t_m^{(l)}(z) = \prod_{j=1}^{m} (z - z_{m,l}^{(j)}) \qquad (z_{m,l}^{(j)} \in E_l)$$

and denote

$$M_m^{(l)} = \max_{z\in E_l} |t_m^{(l)}(z)|.$$

Let $\{m_l\}$ be a sequence of indices such that

$$\lim_{l\to\infty} (M_{m_l}^{(l)})^{1/l^2 m_l} = 0.$$

This is possible, since $\mathrm{Cap}_2 E_l = 0$. Now consider the function

$$\Phi(z, w) = \sum_{l=1}^{\infty} \left(\frac{e\rho}{l^2 m_l}\right)^{l^2 m_l/\rho} w^{l^2 m_l} t_{m_l}^{(l)}(z).$$

This is an entire function of upper order $\bar{\rho}_w = \rho$ and upper type $\bar{\sigma}_w = 1$. Indeed, for all $z \in \{z: |z| < e^l\}$ we have

$$|t_{m_l}^{(l)}(z)| \leqq \prod_{j=1}^{m_l} |z - z_{m_l,l}^{(j)}| < (2e^l)^{m_l}.$$

Consequently in any disk $|z| < R$

$$\overline{\lim_{l\to\infty}} \frac{l^2 m_l}{e\rho}\left\{\left|t_{m_l}^{(l)}(z)\right|\left(\frac{e\rho}{l^2 m_l}\right)^{l^2 m_l/\rho}\right\}^{\rho/l^2 m_l} \leqq \overline{\lim_{l\to\infty}} \{(2e^l)^{m_l}\}^{\rho/l^2 m_l} = 1. \tag{3.2.9}$$

Moreover, the convergence in (3.2.9) is uniform in z in any bounded region of the plane $C_{(z)}^1$. Consequently $\Phi(z, w)$ is an entire function of upper order $\bar{\rho}_w \leqq \sigma$ and upper type $\bar{\rho}_w \leqq 1$.

Now take a point $z^0 \notin \bar{E}$, and let us compute the type $\sigma_\Phi(z^0)$ for order ρ. On the one hand, $\sigma_\Phi(z^0) \leqq \bar{\sigma}_w \leqq 1$. On the other,

$$\sigma_\Phi(z^0) = \overline{\lim_{l\to\infty}} \frac{l^2 m_l}{e\rho}\left\{\left|t_{m_l}^{(l)}(z^0)\right|\left(\frac{e\rho}{l^2 m_l}\right)^{l^2 m_l/\rho}\right\}^{\rho/l^2 m_l} \geqq \overline{\lim_{l\to\infty}} \{\delta^{m_l}\}^{\rho/l^2 m_l} = 1,$$

where $\sigma = \min_{z'\in\bar{E}} |z^0 - z'|$.[15] Consequently $\sigma_\Phi(z^0) = 1$, $\bar{\sigma}_w = 1$, and indeed $\bar{\rho}_w = \rho$. Since the only condition on the point z^0 is $z^0 \notin \bar{E}$, we have thus proved that $N_\sigma(\Phi) \subset \bar{E}$. We now claim that $\sigma_\Phi(z') = 0$ for $z' \in E$. Indeed, let $z' \in E$; then z' is an element of all the sets E_l for sufficiently large l. Therefore

$$\sigma_\Phi(z') = \overline{\lim_{l\to\infty}} \frac{l^2 m_l}{e\rho}\left\{\left(\frac{e\rho}{l^2 m_l}\right)^{l^2 m_l/\rho}\left|t_{m_l}^{(l)}(z')\right|\right\}^{\rho/l^2 m_l} \leqq \overline{\lim_{l\to\infty}} \{M_{m_l}^{(l)}\}^{\rho/l^2 m_l} = 0.$$

Consequently $E \subset N_\sigma(\Phi)$. The proof is thus complete, on the assumption that $\bar{E}$ is not the entire plane.

Now suppose that $\bar{E}$ is the entire plane. Since then the inclusion $N_\sigma(\Phi) \subset \bar{E}$ is obvious, it will suffice to construct an entire function $\Phi(z, w)$ whose exceptional set $N_\sigma(\Phi)$ contains the given set E. To do this, we again use the representation $E = \bigcup_1^\infty E_l$, where $E_l \subset E_{l+1}$, $l = 1,2,\cdots$, and each set E_l is closed and has zero capacity. Enlarge the sets E_l by adding the complex conjugates of all their elements. Denote the resulting sets by A_l. It is obvious that $E \subset A = \bigcup_1^\infty A_l$. Now consider the sets

$$A^{(1)} = A \cap \{z: \operatorname{Im} z \geqq 0\}. \qquad A^{(2)} = A \cap \{z: \operatorname{Im} z \leqq 0\}.$$

It is clear that the sets $A^{(1)}$ and $A^{(2)}$ satisfy the assumptions of the theorem and do not coincide with the entire plane. We can thus construct two entire functions $\Phi_1(z, w)$ and $\Phi_2(z, w)$ such that

$$\begin{aligned} \bar{\rho}_w(\Phi_1) &= \bar{\rho}_w(\Phi_2) = \rho, \\ \bar{\sigma}_w(\Phi_1) &= \bar{\sigma}_w(\Phi_2) = 1, \\ \sigma_{\Phi_i}(z) &= 0 \quad \text{for } z \in A^{(i)}, \qquad i = 1,2, \end{aligned} \tag{3.2.10}$$

and

$$\sigma_{\Phi_i}(z) = 1 \quad \text{for } z \notin A^{(i)}, \qquad i = 1,2.$$

It follows from the construction of the sets $A^{(i)}$ and the functions $\Phi_i(z, w)$, $i = 1, 2$, that

[15] Recall that $z_{m,l}^{(j)} \in E_l$.

$$\overline{\Phi_1(z, w)} = \Phi_2(z, w). \tag{3.2.11}$$

Now take a number t_0 on the real axis in the plane $C^1_{(z)}$ such that both $\sigma_{\Phi_1}(t_0) = 1$ and $\sigma_{\Phi_2}(t_0) = 1$. That this is possible follows from (3.2.10) and the Corollary to Theorem 3.2.2.

Consider the function $f(w) = \Phi_1(t_0, w)$ and its indicator $h_f(\theta)$. Suppose that $h_f(\theta)$ assumes its maximum, equal to $\sigma_{\Phi_1}(t_0)$, at $\theta = \theta_0$. Now consider the function

$$\Phi(z, w) = \Phi_1(z, w)\, \Phi_2(z, e^{-2i\theta_0} w).$$

Since the type of a product of entire functions clearly cannot exceed the sum of types of the factors, we have

$$\bar{\sigma}_w(\Phi) \leqq \bar{\sigma}_w(\Phi_1) + \bar{\sigma}_w(\Phi_2) = 2, \tag{3.2.12}$$

and for $z \in A$

$$\sigma_\Phi(z) \leqq \sigma_{\Phi_1}(z) + \sigma_{\Phi_2}(z) \leqq 1.$$

On the other hand, it follows from (3.2.11) and the fact that t_0 is real that

$$\Phi(t_0, re^{i\theta_0}) = \Phi_1(t_0, re^{i\theta_0})\, \Phi_2(t_0, re^{-i\theta_0}) = \left|\Phi_1(t_0, re^{i\theta_0})\right|^2.$$

Hence in view of the special choice of the number θ_0 we conclude that for any $\varepsilon > 0$ the inequality

$$\left|\Phi(t_0, re^{i\theta_0})\right| \geqq \exp\left(2(h_f(\theta_0) - \varepsilon) r^\rho\right) = \exp\left(2(1 - \varepsilon) r^\rho\right)$$

holds for some sequence of numbers r increasing to infinity. Consequently $\sigma_\Phi(t_0) \geqq 2$. Combined with (3.2.12), this proves that $\bar{\sigma}_w(\Phi) = 2$. Consequently $A \subset N_\sigma(\Phi)$, and since $E \subset A$, it also follows that $E \subset N_\sigma(\Phi)$. This completes the proof of Theorem 3.2.7.

4. *Construction of an entire function with prescribed type* $\sigma^*_f(z)$. In § 2.1 we showed that under certain natural restrictions the regularization $\sigma^*_f(z)$ of the type $\sigma_f(z)$ of an entire function $f(z, w)$, $z \in C^n$, $w \in C^1$, in the variable w satisfies the following condition: $\ln\sigma^*_f(z)$ *is a plurisubharmonic function.* We shall now show that this property is characteristic for types $\sigma^*_f(z)$, i.e. *for any given plurisubharmonic function* $u(z)$ *there is at least one entire function* $f(z)$ *such that* $u(z) = \ln \sigma^*_f(z)$. The proof of this statement is based on the same arguments as the proof (§ 5 of Chapter 2) of the theorem according to which any plurisubharmonic function is a Hartogs function. Indeed, consider the domain

$$G = \{z: \ln|w| + \rho^{-1} u(z) < 0\}$$

in the space C^{n+1}. The function $\ln |w| + \rho^{-1} u(z)$ is plurisubharmonic in C^{n+1}, and so G is a pseudoconvex domain. Therefore, by Oka's Theorem on pseudoconvex domains,[16] there exists a function $\Phi(z, w)$ for which G is a domain of holo-

[16] Concerning Hartogs domains and the properties of functions holomorphic in such domains, see, for example, Bochner and Martin [1], Fuks [2], Šabat [1], or Vladimirov [1].

morphy. On the other hand, G, as is clear from its definition, is a Hartogs domain. Consequently the function $\Phi(z, w)$ has an expansion in G as a Hartogs series

$$\Phi(z, w) = \sum_{m=0}^{\infty} g_m(z) w^m,$$

where $g_m(z)$ are functions holomorphic in C^n. Since G is a domain of holomorphy for $\Phi(z, w)$, it follows from the properties of Hartogs series that

$$\overline{\lim_{z' \to z}} \overline{\lim_{m \to \infty}} (g_m(z'))^{1/m} = e^{u(z)/\rho}. \tag{3.2.13}$$

Now consider the function

$$F(z, w) = \sum_{m=0}^{\infty} \frac{g_m(z)}{\Gamma(1 + m/\rho)} w^m.$$

It is readily seen that $F(z, w)$ is an entire function. It follows from (3.2.13) and the formulas expressing the type and order in terms of the coefficients of the power series expansion that

$$\begin{aligned}(\sigma_F(z) e\rho)^{1/\rho} &= \overline{\lim_{m \to \infty}} m^{1/\rho} \left(\frac{|g_m(z)|}{\Gamma(1 + m/\rho)} \right)^{1/m} = \overline{\lim_{m \to \infty}} \frac{m^{1/\rho} |g_m(z)|^{1/m}}{(m/\rho)^{1/\rho} e^{-1/\rho}} \\ &= (e\rho)^{1/\rho} \overline{\lim_{m \to \infty}} |g_m(z)|^{1/m} \leqq e^{u(z)/\rho} (e\rho)^{1/\rho}.\end{aligned} \tag{3.2.14}$$

Thus the function $F(z, w)$ has finite type $\sigma_F(z)$ in w for order ρ. On the other hand, by (3.2.14) and (3.2.13) we have $\sigma_F^*(z) = e^{u(z)}$, and so $\ln \sigma_F^*(z) = u(z)$. This completes the construction of the required function.

§ 3. Growth of an entire function of finite order and distribution of singularities of the associated function

1. *Entire functions of order* $\rho_f = 1$. Let

$$f(w) = \sum_{m=0}^{\infty} \frac{a_m}{m!} w^m$$

be an entire function of order $\rho_f = 1$ (of one variable w). We define the function associated with $f(w)$ as

$$F(w) = \sum_{m=0}^{\infty} \frac{a_m}{w^{m+1}}.$$

It is well known that the growth of the function $f(w)$ is intimately related to the position of the singularities of $F(w)$. Thus we have

THEOREM 3.3.1 (BOREL). *The type of $f(w)$ is equal to the radius of convergence of the series representing the function $F(w)$.*

The following more precise result is due to Pólya.

THEOREM 3.3.2 (see, for example, Levin [1]). *Let*

$$h_f(\theta) = \overline{\lim_{t\to\infty}} \frac{l}{t} \ln|f(te^{i\theta})|$$

be the indicator of an entire function $f(w)$ of normal type for order $\rho_f = 1$, and let

$$H_B(\theta) = \sup_{w\in B} \operatorname{Re} we^{-i\theta}$$

be the support function of the smallest convex compact set B to whose interior the associated function $F(w)$ is analytically continuable. Then $h_f(\theta) = H_B(-\theta)$.

By analogy with the one-variable case, the associated function of an entire function

$$f(z) = \sum_{\|k\|=0}^{\infty} \frac{a_k}{k!} z^k \qquad (z \in C^n)\ ^{17)} \tag{3.3.1}$$

of order $\rho_f = 1$ is defined to be

$$F_f(z) = \sum_{\|k\|=0}^{\infty} \frac{a_k}{z^{k+1}} = \sum_{\|k\|=0}^{\infty} \frac{a_k}{z_1^{k_1+1} \cdots z_n^{k_n+1}}. \tag{3.3.2}$$

In comparison with the one-variable case, investigation of the relation between the growth of the function $f(z)$ and the distribution of singularities of $F(z)$ is complicated by a number of factors. One of these is that the singular points of the function $F(z)$ cannot all lie in a bounded region of C^n. Nevertheless, the proof of the analog of Borel's theorem is extremely simple:

THEOREM 3.3.3 (RONKIN [1]). *The hypersurface of associated types for associated orders $1, \cdots, 1$*[18] *of an entire function $f(z)$ coincides with the hypersurface of associated radii of convergence of the series defining its associated function $F(z)$.*

PROOF. By (0.6.6), associated radii of convergence $r_1 > 0, \cdots, r_n > 0$ of the series (3.3.2) are defined by

$$\overline{\lim_{\|k\|\to\infty}} (a_k/r^k)^{1/\|k\|} = 1. \tag{3.3.3}$$

On the other hand, associated types $\sigma_1 > 0, \cdots, \sigma_n > 0$ of $f(z)$ for associated orders $1, \cdots, 1$ are defined according to (3.1.28) by

$$\overline{\lim_{\|k\|\to\infty}} (a_k/\sigma_1^{k_1} \cdots \sigma_n^{k_n})^{1/\|k\|} = 1.$$

By (3.3.3) this immediately implies that the hypersurface of associated radii of convergence of the series (3.3.2) coincides with the hypersurface of associated types, as required.

[17] Recall that $k! = k_1! \cdots k_n!$.

[18] Recall that for $\rho_f = 1$ the numbers $1, \cdots, 1$ form a system of associated orders (see § 6 of Chapter 2).

Generalization of Pólya's theorem to several variables is less simple. To accomplish this, as in the case $n = 1$ we first establish an integral representation of the function $f(z)$ in terms of the associated function $F_f(z)$, or, more precisely, its analytic continuation (also denoted by $F_f(z)$), and conversely:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{l_1} \cdots \int_{l_n} F_f(\zeta) e^{\langle z,\zeta\rangle} d\zeta_1 \wedge \cdots \wedge d\zeta_n, \tag{3.3.4}$$

$$F_f(z) = \int_{A_\varphi} e^{-\langle z,\zeta\rangle} f(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n, \tag{3.3.5}$$

where $A_\varphi = \{\zeta : \zeta_j = t_j e^{-i\varphi_j}, t_j > 0, j = 1, \cdots, n\}$ and the simple contours $l_1, \cdots, l_n$ are the boundaries of domains $D_1, \cdots, D_n$ in C^1 such that the function $F_f(z)$ is analytically continuable to some neighborhood of the set $D = \{z: z_j \notin D_j, j = 1, \cdots, n\}$. Moreover, the representation (3.3.5) is valid in every domain

$$G_{\nu,\varphi} = \{z: \operatorname{Re} z_j e^{-i\varphi_j} > \nu_j, i = 1, \cdots, n\},$$

where the numbers $\nu_1, \cdots, \nu_n$ are such that for arbitrary $r_1 \geqq 0, \cdots, r_n \geqq 0$

$$\ln\left|f(r_1 e^{-i\varphi_1}, \cdots, r_n e^{-i\varphi_n})\right| \leqq \nu_1 r_1 + \cdots + \nu_n r_n + C_{\nu,\varphi} \tag{3.3.6}$$

with a suitable constant $C_{\nu,\varphi}$. Indeed,[19] the integral in (3.3.4) is clearly unchanged if each contour l_j is replaced by a circle $|\zeta_j| = R_j$, where $R_1, \cdots, R_n$ are the coordinates of some interior point of the domain of absolute convergence of the series (3.3.2). The series $\sum_{\|k\|=0}^{\infty} a_k e^{\langle z,\zeta\rangle}/\zeta^{k+1}$ is uniformly convergent on the product of the circles $|\zeta_j| = R_j$ and may therefore be integrated term by term there. Thus to prove (3.3.3) we need only point to the obvious equality

$$\int_{|\zeta_j|=R_j} e^{z_j\zeta_j} \zeta_j^{-k_j-1} d\zeta_j = \frac{z_j^{k_j}}{k_j!}.$$

To prove (3.3.5), which defines a certain analytic continuation of the function represented by the series (3.3.2), we first observe that the integral in (3.3.5) converges absolutely and uniformly in any domain

$$\{z: \operatorname{Re} z_j e^{-i\varphi_j} > \nu_j + \varepsilon, j = 1, \cdots, n\},$$

where $\varepsilon > 0$ and $\nu_1, \cdots, \nu_n$ are numbers satisfying (3.3.6). This follows from the estimate

$$\int_{A_\varphi} \left|e^{-\langle z,\zeta\rangle} f(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n\right| \leqq e^{C_{\nu,\varphi}} \prod_{j=1}^{n} \int_0^\infty \exp\{t_j(\nu_j - \operatorname{Re} z_i e^{-i\varphi_j})\} dt_j$$

$$\leqq e^{C_{\nu,\varphi}} \prod_{j=1}^{n} \int_0^\infty e^{-\varepsilon t_j} dt_j < \infty.$$

Consequently the integral in question represents a function analytic in the domain

[19] The proof of (3.3.4) and (3.3.5) is completely analogous to that of the corresponding representations in the case $n = 1$.

$G_{\nu,\varphi}$, and therefore to prove (3.3.5) for this domain it will suffice to show that it holds in the following domain G' which lies in the intersection of $G_{\nu,\varphi}$ with the domain of convergence of the series (3.3.2):

$$G' = \{z: \operatorname{Re} z_j e^{-i\varphi_j} > 3\sigma_j + 1, j = 1, \cdots, n\},$$

where $\sigma_1, \cdots, \sigma_n$ is a system of associated types of the function $f(z)$. For $z \in G'$,

$$\left|\exp\left(-z_j t_j e^{-i\varphi_j}\right)\right| < \exp(-3\sigma_j t_j - t_j).$$

On the other hand,

$$\frac{|a_k|}{k!} \leqq \frac{M_f(r)}{r^k} \leqq \frac{C_\varepsilon \exp\left(\sum_1^n (\sigma_j + \varepsilon) r_j\right)}{r^k}.$$

Setting $r_j = 2t_j$, we see further that, for $z \in G'$ and $\zeta \in A_\varphi$, $\left|(a_k/k!)e^{-\langle z,\zeta\rangle}\zeta_k\right| \leqq 2^{-\|k\|} C_\varepsilon \exp\left(-\sum_1^n t_j(\sigma_j + 1 - 2\varepsilon)\right)$, which immediately implies that the series $\sum_{\|k\|=0}^{\infty} (a_k/k!)e^{-\langle z,\zeta\rangle}\zeta_k$ is uniformly convergent for $\zeta \in A_\varphi$ and $z \in G'$. Integrating this series term by term over the domain A_φ, recalling the while that

$$e^{-i(m+1)\varphi_j} \int_0^\infty e^{-z_j t_j e^{-i\varphi_j}} t_j^m \, dt_j = \frac{m!}{z_j^{m+1}},$$

we see that the representation (3.3.5) is valid.

Following Ivanov [1], we now introduce the set $C_f(\varphi) = C_f(\varphi_1, \cdots, \varphi_n)$ of all points $\nu \in R^n$ such that the function $F_f(z)$ is analytically continuable from a neighborhood of the point $(\infty, \cdots, \infty)$ into the corresponding domain $G_{\nu,\varphi}$. Note that by (3.3.5) the function $F_f(z)$ is analytically continuable as a univalent analytic function to the union

$$\bigcup_{\varphi \in R^n} \bigcup_{\nu \in C_f(\varphi)} G_{\nu,\varphi}.$$

This follows from the observation that the intersection of any two domains $G_{\nu,\varphi}$ and $G_{\nu',\varphi'}$ is simply connected and contains points in any neighborhood of $(\infty, \cdots, \infty)$. It is readily seen that the sets $C_f(\varphi)$ are closed.

Now consider the set $T_f(\varphi)$ of all points $\nu \in R^n$ such that for some constant $C_{\nu,\varphi}$

$$\ln\left|f(r_1 e^{i\varphi_1}, \cdots, r_n e^{i\varphi_n})\right| \leqq C_{\nu,\varphi} + \nu_1 r_1 + \cdots + \nu_n r_n \qquad \forall r \in R_+^n.$$

This set characterizes the growth of $f(r_1 e^{i\varphi_1}, \cdots, r_n e^{i\varphi_n})$ in the variables $r_1, \cdots, r_n$ for fixed $\varphi_1, \cdots, \varphi_n$. The construction of the set $T_f(\varphi)$ in terms of $\ln\left|f(r_1 e^{i\varphi_1}, \cdots, r_n e^{i\varphi_n})\right|$ is exactly the same as that of the set $B_\sigma(f)$ in terms of $\ln M_f(r)$.

EXAMPLE. Let

$$f(z_1, z_2) = \sum_{m=0}^{\infty} (-1)^m \frac{(z_1 z_2)^m}{(2m)!} = \cos\sqrt{z_1 z_2}.$$

Then

$$\begin{aligned}\left|f(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2})\right| &= \left|\cos\left(\sqrt{r_1 r_2}\exp\left(i(\varphi_1 + \varphi_2)/2\right)\right)\right| \\ &= \frac{1}{2}\exp\left(\sqrt{r_1 r_2}\sin\left((\varphi_1 + \varphi_2)/2\right)\right) + o(1) \quad (|r| \to \infty).\end{aligned}$$

It is readily seen that the curve of associated types of the function $\sqrt{r_1 r_2} \in \mathfrak{A}$ for conjugate orders 1, 1 is the curve $\sigma_1 \sigma_2 = 1/4$. Consequently

$$\bar{T}_f(\varphi_1, \varphi_2) = \left\{(\nu_1, \nu_2): \nu_1, \nu_2 \geqq \frac{1}{4}\left|\sin\frac{\varphi_1 + \varphi_2}{2}\right|, \nu_1 \geqq 0, \nu_2 \geqq 0\right\}.$$

The following theorem is a natural analog of Pólya's theorem for several variables.

THEOREM 3.3.4.[20] *Let $f(z)$ be an entire function of at most normal type for order $\rho_f = 1$. Then $\bar{T}_f(\varphi) = C_f(-\varphi)$ for all $\varphi \in R^n$.*

The proof of this theorem depends essentially on the following lemmas.

LEMMA 3.3.1. *Let the function $g(z)$ be analytic in the polydisk $D = \{z: |z_i| < R_i, i = 1, \cdots, n\}$. Let $z^0 = (z_1^0, \cdots, z_n^0)$ be a singular point of $g(z)$ on the boundary of the polydisk, and P the set of all values of i such that $|z_i^0| < R_i$. Then all points of the manifold*

$$D_0 = \{z: z_i = z_i^0, |z_i| \underset{i \in P}{<} R_i\}$$

are singular points of $g(z)$.

PROOF. Set

$$d(z, \zeta) = \max_{1 \leqq i \leqq n} |z_i - \zeta_i|, \qquad z \in C^n, \ \zeta \in C^n.$$

Let $\Delta_g(z)$, $z \in D$, denote the radius R of the largest polydisk $\{\zeta: |\zeta_j - z_j| < R, j = 1, \cdots, n\}$, in which the function $g(\zeta)$ is analytically continuable. In other words,

$$\Delta_g(z) = \inf_{\zeta \in \Lambda_g} d(z, \zeta),$$

where Λ_g is the set of singular points of $g(z)$. The function $\Delta_g(z)$ is clearly continuous and positive in D. It also follows from the definition of $\Delta_g(z)$ that it coincides with the largest value of R such that the series

$$\sum_{\|k\|=0}^{\infty} \frac{1}{k!} D^k g(z) \cdot (\zeta - z)^k,$$

where $D^k = \partial^{\|k\|}/\partial z_1^{k_1} \cdots \partial z_n^{k_n}$, converges in the polycylinder $\{\zeta: |\zeta_i - z_i| < R, i = 1, \cdots, n\}$. Hence by (0.6.3) it follows that

$$\frac{1}{\Delta_g(z)} = \overline{\lim_{\|k\| \to \infty}} \left(|D^k g(z)|/k!\right)^{1/\|k\|}.$$

[20] This theorem was first proved for $n = 2$ by Ivanov [1], [2]. For $n > 2$ it was proved by Stavskiĭ [1],[2]. This proof is due to Ronkin [3].

The functions appearing on the right of this relation are plurisubharmonic, and therefore the regularization of $\Delta_g^{-1}(z)$ is also a plurisubharmonic function. But since $\Delta_g(z)$ is continuous and positive, the regularization of $\Delta_g^{-1}(z)$ is the function itself, and hence $\Delta_g^{-1}(z)$ is a plurisubharmonic function.

Let $i \in P$ and $t < \min_i R_i$; consider the disk $D_{i,t} = \{z_i\colon |z_i| < R_i - t\}$. For $i \notin P$ set $z_i^{(t)} = z_i^0(1 - t/R_i)$.

Let D_t denote the set of points z such that $z_i = z_i^{(t)}$ for $i \notin P$, and $z_i \in D_{i,t}$ for $i \in P$. Since the function $\Delta_g^{-1}(z)$ is plurisubharmonic in D, it is also plurisubharmonic as a function of the variables z_i, $i \in P$, on the manifold D_t. Since plurisubharmonic functions satisfy the maximum principle, it follows that $\Delta_g(z)$ satisfies the minimum principle on D_t, i.e. it can assume its minimum on D_t at an interior point only if

$$\Delta_g(z) = \min_{z \in D_t} \Delta_g(z) \qquad \forall z \in D_t.$$

It follows from the definition of the manifold D_t and from the fact that $g(z)$ is holomorphic in D that $\Delta_g(z) \geqq t$ $\forall z \in D_t$. On the other hand, we have $\Delta_g(z) = t$ for at least one interior point of D_t. Indeed, the point z' with coordinates z_i' defined by $z_i' = z_i^t$ for $i \in P$ and $z_i' = z_i^0$ for $i \notin P$ is obviously in D_t and satisfies the condition $d(z', z^0) = t$. Hence by the minimum principle we conclude that $\Delta_g(z) = t$ for $z \in D_t$. This means that for any $\varepsilon > 0$ every polydisk $\{\zeta\colon |z_i - \zeta_i| < t + \varepsilon, i = 1, \cdots, n\}$, $z \in D_t$, contains singular points of the function $g(z)$. Letting t tend to zero and noting that $D_t \to D_0$ as $t \to 0$, we see further that any arbitrarily small neighborhood of each point of the set $\bar{D}_0$ contains singular points of the function $g(z)$. Consequently every point of this set is singular. Q.E.D.

LEMMA 3.3.2. *Let $l_j = l_j(R, \varepsilon, \varphi_j)$ be the closed contour in the z_j-plane formed by an arc of the circle $|z_j| = R$ and a segment of the straight line* $\operatorname{Re} z_j e^{-i\varphi_j} = \nu_j + \varepsilon$. *Let $\nu \in C_f(\varphi)$. Then for any $\varepsilon > 0$ there exists a number $R(\varepsilon)$ such that the associated function $F_f(z)$ is analytic at all points z with the following property: for any j the coordinate z_j is either on the contour $l_j(R(\varepsilon), \varepsilon, \varphi_j)$ or outside it.*

PROOF. Let $D_{\varepsilon,j}$ denote the disk into which the half-plane $\{z_j\colon \operatorname{Re} z_j e^{-i\varphi_j} > \nu_j + \varepsilon\}$ is mapped by the function $\zeta_j = 1/z_j$. In other words,

$$D_{\varepsilon,j} = \left\{\zeta_j\colon \left|\zeta_j - \frac{e^{-i\varphi_j}}{2(\nu_j + \varepsilon)}\right| < \frac{1}{2(\nu_j + \varepsilon)}\right\}.$$

Set $D_\varepsilon = \{\zeta\colon \zeta_j \in D_{\varepsilon,j}, j = 1, \cdots, n\}$ and $g(\zeta) = F(1/\zeta_1, \cdots, 1/\zeta_n)$. Since for sufficiently large $R > 0$ the function $F(z)$ is holomorphic in the polycylinder $\{z\colon |z_j| > R, j = 1, \cdots, n\}$, it follows that the function $g(\zeta)$ is holomorphic in some neighborhood of the origin. Since ν is in $C_f(\varphi)$, it follows that the function $F_f(z)$ is holomorphic in the product of the half-planes $\operatorname{Re} z_j e^{-i\varphi_j} > \nu_j + \varepsilon$, and therefore the function $g(\zeta)$ is holomorphic in D_ε. Each point ζ in a domain D_{ε_1} clearly lies also

in the closure of any domain D_{ε_2} such that $\varepsilon_2 > \varepsilon_1 > 0$. Hence the function $g(\zeta)$ is holomorphic at all points of the set $\bar{D}_\varepsilon$, except perhaps for points ζ which have at least one coordinate equal to zero. Consequently, if $\bar{D}_\varepsilon$ contains a singular point ζ^0 of $g(\zeta)$, then the set Q of all j such that $\zeta_j^0 = 0$ is not empty. By Lemma 3.3.1 we conclude that all points of the manifold

$$\{\zeta: \zeta_j = \zeta_j^0, \zeta_j \in D_{\varepsilon,j}\}_{j\in Q}$$

are singular points of $g(\zeta)$. This contradicts the fact that $g(\zeta)$ is holomorphic at the point $(0, \cdots, 0)$. Consequently there are no singular points of $g(\zeta)$ in $\bar{D}_\varepsilon$. Now, as is well known, any function holomorphic on a compact set is holomorphic in some neighborhood of this set. Thus for sufficiently small $\delta = \delta(\varepsilon) > 0$ the function $g(\zeta)$ is holomorphic in the domain

$$D' = \Big\{\zeta: \zeta_j \in D_{\varepsilon/2,j} \bigcup_{j=1,\cdots,n} \{\zeta_j: |\zeta_j| < \delta\}\Big\}$$

and consequently the function $F_f(z)$ is holomorphic in the cartesian product of the domains

$$D_j'' = \{z_j: \operatorname{Re} z_j e^{-i\varphi_j} > \nu_j + \varepsilon/2\} \cup \{z_j: |z_j| > 1/\delta\}.$$

To complete the proof of the lemma, it is now sufficient to note that for any $R > 1/\delta(\varepsilon)$ the contour $l_j(R, \varepsilon, \varphi_j)$ lies in the domain D_j''. Q.E.D.

We now proceed to the proof of Theorem 3.3.4. Previously, in the proof of (3.3.5), we showed that, for any system of numbers $\nu_1, \cdots, \nu_n$ such that for any $r_1 \geqq 0, \cdots, r_n \geqq 0$

$$\ln|f(r_1 e^{-i\varphi_1}, \cdots, r_n e^{-i\varphi_n})| \leqq \nu_1 r_1 + \cdots + \nu_n r_n + C_{\nu,\varphi},$$

the function $F_f(z)$ is analytic in the domain

$$G_{\nu,\varphi} = \{z: \operatorname{Re} z_j e^{-i\varphi_j} > \nu_j, j = 1, \cdots, n\}.$$

Consequently every point ν in the set $T_f(-\varphi)$ is also in the set $C_f(\varphi)$, i.e. we have

$$T_f(-\varphi) \subset C_f(\varphi). \tag{3.3.7}$$

Conversely, let $\nu \in C_f(\varphi)$. It follows from Lemma 3.3.2 that we can take the contours l_j in (3.3.4) to be the contours $l_j(R(\varepsilon), \varepsilon, \varphi_j)$, and then the function $f(z)$ can be estimated as follows:

$$|f(r_1 e^{-i\varphi_1}, \cdots, r_n e^{-i\varphi_n})| = \left|\frac{1}{(2\pi)^n}\int_{l_1}\cdots\int_{l_n} F_f(\zeta)\, e^{\langle z,\zeta\rangle}\, d\zeta_1 \wedge \cdots \wedge d\zeta_n\right|$$

$$\leqq \text{const}\cdot\prod_{j=1}^{n} \max_{\zeta_j\in l_j} \exp(r_j \operatorname{Re} e^{-i\varphi_j}\zeta_j) = \text{const}\cdot\exp\Big(\sum_1^n(\nu_j + \varepsilon)r_j\Big).$$

It follows from this estimate that for any $\varepsilon > 0$ we have $(\nu_1 + \varepsilon, \cdots, \nu_n + \varepsilon) \in T_f(-\varphi)$, whenever $(\nu_1, \cdots, \nu_n) \in C_f(\varphi)$. Consequently

$$C_f(\varphi) \subset \bar{T}_f(-\varphi). \tag{3.3.8}$$

Since the set $C_f(\varphi)$ is closed, it follows from (3.3.7) and (3.3.8) that $C_f(\varphi) = \bar{T}_f(-\varphi)$. This completes the proof.

We now consider a few properties of the sets $C_f(\varphi)$ and $T_f(\varphi)$.

(a) *The sets $\bar{T}_f(\varphi)$ and $C_f(\varphi)$ are convex.*

Indeed, let $\nu \in T_f(\varphi)$ and $\nu' \in T_f(\varphi)$. Then, by the definition of the set $T_f(\varphi)$, for any $r_1, \cdots, r_n$ we have

$$\ln\left|f(r_1e^{i\varphi_1}, \cdots, r_ne^{i\varphi_n})\right| \leqq \nu_1 r_1 + \cdots + \nu_n r_n + C_{\nu,\varphi},$$
$$\ln\left|f(r_1e^{i\varphi_1}, \cdots, r_ne^{i\varphi_n})\right| \leqq \nu_1' r_1 + \cdots + \nu_n' r_n + C_{\nu',\varphi}.$$

Consequently, for any $\lambda \in [0, 1]$ and $\mu = 1 - \lambda$, and all $r_1, \cdots, r_n$,

$$\ln\left|f(r_1e^{i\varphi_1}, \cdots, r_ne^{i\varphi_n})\right| \leqq \sum_{j=1}^{n} (\lambda\nu_j + \mu\nu_j')r_j + \lambda C_{\nu,\varphi} + \mu C_{\nu',\varphi},$$

whence it follows that any point on the segment joining ν and ν' also lies in $T_f(\varphi)$. Thus $T_f(\varphi)$ is convex, and therefore so is its closure $\bar{T}_f(\varphi)$. The convexity of $C_f(\varphi)$ now follows from Theorem 3.3.4.

(b) *The sets $\bar{T}_f(\varphi)$ and $C_f(\varphi)$ are independent of the choice of the origin in the space C^n in which the function $f(z)$ is defined.*

Indeed, any displacement of the origin in C^n means that the function $f(z)$ must be replaced by $f_a(z) = f(z + a)$ $(a \in C^n)$. Let us estimate $f_a(z)$. For $\nu \in C_f(-\varphi)$ and $l_j = l_j(R(\varepsilon), \varepsilon, -\varphi_j)$, $j = 1, \cdots, n$, we have

$$|f_a(z)| = \left|f(a_1 + r_1e^{i\varphi_1}, \cdots, a_n + r_ne^{i\varphi_n})\right| = \left(\frac{1}{2\pi}\right)^n \left|\int_{l_1}\cdots\int_{l_n} F(\zeta)e^{\langle z+a,\zeta\rangle}\, d\zeta_1 \wedge \cdots \wedge d\zeta_n\right|$$

$$\leqq \text{const} \cdot \max_{\substack{\zeta_j \in l_j \\ j=1,\cdots,n}} \left|e^{\langle z+a,\zeta\rangle}\right| \leqq \text{const} \cdot \prod_{j=1}^{n} \exp\left\{r_j \max_{\zeta_j \in l_j} \operatorname{Re} e^{i\varphi_j}\zeta_j + \max_{\zeta_j \in l_j} \operatorname{Re} a_j\zeta_j\right\}$$

$$= \text{const} \cdot \exp\left((\nu_1 + \varepsilon)r_1 + \cdots + (\nu_n + \varepsilon)r_n\right).$$

Consequently $(\nu_1 + \varepsilon, \cdots \nu_n + \varepsilon) \in T_{f_a}(\varphi)$, and so $\nu \in \bar{T}_{f_a}(\varphi)$. This proves that $C_f(-\varphi) \subset \bar{T}_{f_a}(\varphi)$, and also $\bar{T}_f(\varphi) \subset \bar{T}_{f_a}(\varphi)$. Similarly, taking $f_a(z)$ as the original function, we show that $\bar{T}_{f_a}(\varphi) \subset \bar{T}_f(\varphi)$. Consequently $\bar{T}_f(\varphi) = \bar{T}_{f_a}(\varphi)$, i.e. the set $\bar{T}_f(\varphi)$ is invariant under translations of the space C^n, and hence so is $C_f(\varphi)$.

(c) *The intersection $\{\nu: -\nu \in C_f(\varphi + \pi)\} \cap C_f(\varphi)$ contains no interior points for any $\varphi \in R^n$.*

Indeed, suppose (to fix ideas) that $\varphi_1 = \pi/2, \cdots, \varphi_n = \pi/2$, and let ν^0 be an interior point of the above intersection. Then for sufficiently small $\varepsilon > 0$ the function $F_f(z)$ is analytic in the domains $\{z: y_i > \nu_i^0 - \varepsilon, i = 1, \cdots, n\}$ $(y_i = \operatorname{Im} z_i)$ and $\{z: y_i < \nu_i^0 + \varepsilon, i = 1, \cdots, n\}$, and hence also in their union, which is a tube domain[21] with base

[21] A domain $T_B \subset C^n$ is said to be a tube domain with base B if B is a domain in $R^n_{(y)}$ and $T_B = \{z: z = x + iy \in C^n, y \in B, x \in R^n\}$.

$$B = \{y: y \in R^n, \underset{i=1,\cdots,n}{y_i < \nu_i^0 + \varepsilon}\} \cup \{y: y \in R^n, \underset{i=1,\cdots,n}{y_i > \nu_i^0 - \varepsilon}\}.$$

It is known [22] that a function holomorphic in a tube domain with base B is analytically continuable to the tube domain whose base is the convex hull of B. In the present case, the convex hull of B is the entire space R^n. Thus $F_f(z)$ is an entire function, which is impossible. Thus the above intersection cannot contain any interior points.

In many cases Theorem 3.3.4 enables one to reduce investigation of the growth of an entire function $f(z)$ to investigation of the distribution of singularities of its associated function $F(z)$, which is sometimes simpler. The following proposition is an example of such an application of Theorem 3.3.4.

THEOREM 3.3.5 (RONKIN [14]).[23] *Let $f(z)$ be an entire function of at most normal type for order $\rho_f = 1$. Assume, moreover, that for any $\varepsilon > 0$ there exist constants $C_1^{(\varepsilon)} > 0$ and $C_2^{(\varepsilon)} > 0$ such that*

$$|f(x)| \leqq C_1^{(\varepsilon)} e^{\varepsilon|x|} \qquad \forall x \in R^n \tag{3.3.9}$$

and

$$|f(iy)| \leqq C_2^{(\varepsilon)} \exp\Big(\sum_{j=1}^{n} (\sigma_j + \varepsilon)|y_j|\Big). \tag{3.3.10}$$

Then for any $\varepsilon > 0$ there exists a constant $C_3^{(\varepsilon)}$ such that for all $z = x + iy \in C^n$

$$|f(z)| \leqq C_3^{(\varepsilon)} \exp\Big\{\sum_{j=1}^{n} (\sigma_j + \varepsilon)|y_j| + \varepsilon|x|\Big\}.$$

PROOF. By Theorem 3.3.4 and property (c) of the sets $C_f(\varphi)$ it follows from (3.3.9) that for any point $\varphi = (\varphi_1, \cdots, \varphi_n)$ whose coordinates take the values 0 and π only, the set $C_f(\varphi)$ is given by

$$C_f(\varphi) = \{\nu: \nu_f \geqq 0, j = 1, \cdots, n.\}$$

Consequently z can be a singular point of the associated function $F_f(z)$ only if $\operatorname{Re} z_j = 0$ for at least one value of j. Further, by the same theorem it follows from (3.3.10) that, for any point φ whose coordinates take only the values $\pi/2$ and $-\pi/2$,

$$\{\nu: \nu_j \geqq \sigma_j, j = 1, \cdots, n\} \subset C_f(\varphi).$$

Consequently no point z such that $|y_j| > \sigma_j$ for all j can be a singular point of $F_f(z)$. We claim that $F_f(z)$ has a holomorphic continuation to the cartesian product of the z_j-planes, cut along the segments $-\sigma_j \leqq y_j \leqq \sigma_j$ of the imaginary axis — call this domain D. Indeed, it follows from previous statements concerning the

[22] See for example, Bochner and Martin [1], Vladimirov [1], Šabat [1] or Hörmander [2].

[23] A similar theorem may be found in Martineau [1] (cf. Proposition 1.10). Martineau's theorem differs from Theorem 3.3.5 in that instead of (3.3.10) one assumes that

$$|f(z)| \leqq C_2^{(\varepsilon)} \exp((\sigma_1 + \varepsilon)|z_1| + \cdots + (\sigma_n + \varepsilon)|z_n|) \qquad \forall z \in C^n.$$

singular points of $F_f(z)$ that a point z^0 in D can be singular only if the set P of all j such that $\operatorname{Re} z_j^0 = 0$ and $\operatorname{Im} z_j^0 > \sigma_j$ is not empty. Suppose that one of these points is indeed singular. Consider the polydisk (Figures 4 and 5)

$$\{z: |z_j - z_j^0 - 1| < 1, |z_j - i(1 + \sigma_j) - R \operatorname{sign} x_j^0| < R\},$$

where R is so chosen that for $j \notin P$

$$|z_j^0 - i(\sigma_j + 1) - R \operatorname{sign} x_j^0| < R.$$

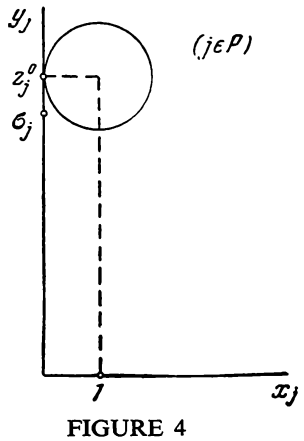

FIGURE 4

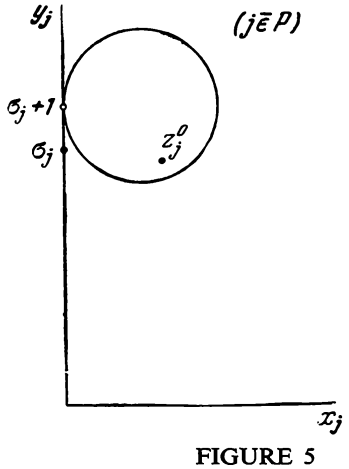

FIGURE 5

The function $F_f(z)$ is clearly holomorphic in this polycylinder, and the point z^0 is a boundary point. Thus the assumptions of Lemma 3.3.1 are satisfied; it follows that all points z such that $z_j = z_j^0$ for $j \in P$ and $|z_j - i(1 + \sigma_j) - R \operatorname{sign} x_j^0| < R$ are singular points of $F_f(z)$. In particular, the point z with coordinates $z_j = z_j^0$ for $j \in P$ and $z_j = i(\sigma_j + 1)$ for $j \notin P$ is singular; this contradicts the fact that the function $F_f(z)$ is holomorphic at points z such that $|\operatorname{Im} z_j| > \sigma_j$, $j = 1, \cdots, n$. Consequently $F_f(z)$ is holomorphic in the domain D.

Now let the contours l_j in the representation (3.3.4) be the boundaries of rectangles $\{z_j: |x_j| < \varepsilon, |y_j| < \sigma_j + \varepsilon\}$. This is possible, since $F_f(z)$ is holomorphic in D. The function $f(z)$ then satisfies the following estimate:

$$|f(z)| = \left(\frac{1}{2\pi}\right)^n \left| \int_{l_1} \cdots \int_{l_n} e^{\langle z,\zeta\rangle} F(\zeta)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n \right| \leqq \text{const} \cdot \max_{\substack{\zeta_j \in l_j \\ j=1,\cdots,n}} e^{\operatorname{Re}\langle z,\zeta\rangle}$$

$$= \text{const} \cdot \exp\left\{ \sum_{j=1}^n (\sigma_j + \varepsilon)|y_j| + \varepsilon \sum_{j=1}^n |x_j| \right\}.$$

This completes the proof.

2. *Entire functions of arbitrary order.* Results analogous to those of the preceding subsection can be established for entire functions of arbitrary order. Before stating these results, we present a theorem dealing with the distribution of singularities of

analytic functions. This theorem will be used in place of Lemma 3.3.1 (which it generalizes) in our study of entire functions of arbitrary order.

THEOREM 3.3.6 (RONKIN [8]). *Let $g(z)$ be an analytic function in the polycylinder $D = \{z: z_i \in D_i,\ i = 1, \cdots, n\}$, where each D_i is a domain in the z_i-plane, $i = 1, \cdots, n$. Let z^* be a boundary point of this polycylinder which is a singular point for $g(z)$, and assume moreover that the set P of values of i such that $z_i^* \in D_i$ is not empty. Then all points of the manifold*

$$M_0 = \left\{z: \underset{i\notin P}{z_i = z_i^*},\ \underset{i\in P}{z_i \in D_i}\right\}$$

are singular points of the function $g(z)$.

PROOF. We first prove a lemma.

LEMMA 3.3.3. *Let the function $u(z)$ be plurisubharmonic in some neighborhood of the closure of the polycylinder $G = \{z: z_i \in G_i,\ i = 1, \cdots, n\}$, where each G_i is a domain in the z_i-plane. Let z' be a point such that $z_i' \in \partial G_i$, $i = 1, \cdots, n$. Then*

$$\overline{\lim_{z\to z', z\in G}}\ u(z) = u(z').$$

PROOF. Since $u(z)$ is plurisubharmonic in a neighborhood of z', it follows that

$$\overline{\lim_{z\to z'}}\ u(z) = u(z').$$

Therefore

$$\overline{\lim_{z\to z', z\in G}}\ u(z) \leqq u(z'). \tag{3.3.11}$$

On the other hand,

$$\overline{\lim_{z\to z', z\in G}}\ u(z) \geqq \overline{\lim_{\substack{z_n\to z_n'\\ z_n\in G_n}}} \cdots \overline{\lim_{\substack{z_1\to z_1'\\ z_1\in G_1}}}\ u(z_1, \cdots, z_n). \tag{3.3.12}$$

It follows from the criterion of § 4.4, Chapter 1, that the domain G_i is thick at each boundary point. Hence, in view of the plurisubharmonicity of $u(z)$ in each variable z_j, we get

$$\overline{\lim_{\substack{z_n\to z_n'\\ z_n\in G_n}}} \cdots \overline{\lim_{\substack{z_1\to z_1'\\ z_1\in G_1}}}\ u(z) = u(z'). \tag{3.3.13}$$

It follows from (3.3.11), (3.3.12) and (3.3.13) that

$$\overline{\lim_{\substack{z\to z'\\ z\in G}}}\ u(z) = u(z').$$

Q.E.D.

REMARK. Let us say that a set $E \subset C^n$ is *C^n-thin at a point z'* if either $z' \notin \bar{E}$ or there exists a function $\Phi(z)$, plurisubharmonic in a neighborhood of z', such that

$$\overline{\lim_{z \to z', z \in E}} \Phi(z) < \Phi(z').$$

Thus Lemma 3.3.3 gives a sufficient condition for a set E to be C^n-thick at a point z'.

For convenience, let us denote the variables z_i for which $i \notin P$ by $\zeta_1, \cdots, \zeta_l$, and those for which $j \in P$ by $\lambda_1, \cdots, \lambda_{n-l}$. Correspondingly we denote the domains D_i by $D_1', \cdots, D_l'$ and $D_1'', \cdots, D_{n-l}''$. We also denote

$$\zeta = (\zeta_1, \cdots, \zeta_l), \qquad \lambda = (\lambda_1, \cdots, \lambda_{n-l}), \qquad z = (\zeta, \lambda);$$
$$D' = \{\zeta : \zeta_i \in D_i', i = 1, \cdots, l\},$$
$$D'' = \{\lambda : \lambda_i \in D_i'', i = 1, \cdots, n - l\},$$

and let $\Delta_g(\zeta^0, \lambda^0)$ denote the largest number t such that the function $g(z) = g(\zeta, \lambda)$ is analytic in ζ and λ at all points of the manifold

$$\{(\zeta, \lambda) : \zeta = \zeta^0, |\lambda_i - \lambda_i^0| < t, i = 1, \cdots, n - l\}.$$

The function $\Delta_g(\zeta, \lambda)$ is obviously defined in any domain in which $g(\zeta, \lambda)$ is holomorphic, and, as we shall presently show, is related to the function $g(\zeta, \lambda)$ by

$$-\ln \Delta_g(\zeta^0, \lambda^0) = \overline{\lim_{\zeta \to \zeta^0}} \overline{\lim_{\|k\| \to \infty}} \frac{1}{\|k\|} \ln |a_k(\zeta, \lambda^0)|, \tag{3.3.14}$$

where $k = (k_1, \cdots, k_{n-l})$ and

$$a_k(\zeta, \lambda^0) = \frac{1}{k!} \frac{\partial^{\|k\|} g(\zeta, \lambda)}{\partial \lambda_1^{k_1} \cdots \partial \lambda_{n-l}^{k_{n-l}}}\bigg|_{\lambda = \lambda^0}.$$

Indeed, in a neighborhood of each point (ζ, λ^0) at which the function $g(\zeta, \lambda)$ is holomorphic, we have

$$g(\zeta, \lambda) = \sum_{\|k\| = 0}^{\infty} a_k(\zeta, \lambda^0)(\lambda - \lambda^0)^k, \tag{3.3.15}$$

and the radius of the largest polydisk $\{\lambda : |\lambda_i - \lambda_i^0| < R(\zeta, \lambda^0), i = 1, \cdots, n - l\}$ in which this series converges is defined by

$$\frac{1}{R(\zeta, \lambda^0)} = \overline{\lim_{\|k\| \to \infty}} |a^k(\zeta, \lambda^0)|^{1/\|k\|}.$$

Consequently $g(\zeta, \lambda)$ is not analytic in the variables $\lambda_1, \cdots, \lambda_{n-l}$ in any polycylinder $\{\lambda : |\lambda_i - \lambda_i^0| < R(\zeta, \lambda^0) + \varepsilon, i = 1, \cdots, n - l\}$, where $\varepsilon > 0$, and so

$$-\ln \Delta_g(\zeta, \lambda^0) \geqq -\ln R(\zeta, \lambda^0) = \overline{\lim_{\|k\| \to \infty}} \frac{1}{\|k\|} \ln |a_k(\zeta, \lambda^0)|. \tag{3.3.16}$$

On the other hand, it follows from Hartogs' Lemma (see § 1 of Chapter 1) that for any $\varepsilon > 0$ the inequality

$$\frac{1}{\|k\|}\ln|a_k(\zeta,\lambda^0)| \leqq \overline{\lim_{\zeta\to\zeta^0}}\ \overline{\lim_{\|k\|\to\infty}}\ \frac{1}{\|k\|}\ln|a_k(\zeta,\lambda^0)| + \varepsilon$$

holds for sufficiently large $\|k\|$ throughout some neighborhood $V = V(\varepsilon, \zeta^0, \lambda^0)$ of the point ζ^0. Hence the series (3.3.15) is uniformly convergent in the domain

$$\left\{(\zeta,\lambda)\colon \zeta\in V, |\lambda_i-\lambda_i^0| < \left(\overline{\lim_{\zeta\to\zeta^0}}\ \overline{\lim_{\|k\|\to\infty}}\ e^{\varepsilon}|a_k(\zeta,\lambda^0)|^{1/\|k\|}\right)^{-1}\right\}$$

and consequently its sum $g(\zeta, \lambda)$ is an analytic function in this domain. It follows that

$$\Delta_g(\zeta^0,\lambda^0) \geqq e^{-\varepsilon}\left(\overline{\lim_{\zeta\to\zeta^0}}\ \overline{\lim_{\|k\|\to\infty}}\ |a_k(\zeta,\lambda^0)|^{1/\|k\|}\right)^{-1}.$$

Further, in view of the fact that $\varepsilon > 0$ is arbitrary, we conclude that

$$\Delta_g(\zeta^0,\lambda^0) \geqq \overline{\lim_{\zeta\to\zeta^0}}\ \overline{\lim_{\|k\|\to\infty}}\ |a_k(\zeta,\lambda^0)|^{1/\|k\|}. \tag{3.3.17}$$

Inequalities (3.3.17) and (3.3.16) imply (3.3.14).

It follows from the representation (3.3.14) and from the properties of plurisubharmonic functions (Chapter 2, § 1) that the function $-\ln\Delta_g(\zeta,\lambda)$ is plurisubharmonic in any domain in which the function $\Delta_g(\zeta,\lambda)$ is defined, i.e. in which the function $g(\zeta,\lambda)$ is holomorphic. [24]

Now assume that the statement of the theorem is false, i.e. the manifold $\{(\zeta,\lambda)\colon \zeta=\zeta^*, \lambda\in D''\}$ contains points of holomorphy of the function $g(\zeta,\lambda)$. Then there are a singular point $(\zeta^*,\lambda^{(1)})$ and a point of holomorphy $(\zeta^*,\lambda^{(2)})$ on this manifold such that for some $\varepsilon > 0$

$$|\lambda_i^{(1)} - \lambda_i^{(2)}| \leqq \varepsilon \qquad (i = 1,\cdots,n-l), \tag{3.3.18}$$

$$\min_{\lambda_i'\in\partial D_i''} |\lambda_i - \lambda_i^{(2)}| \geqq 2\varepsilon. \tag{3.3.19}$$

If follows from (3.3.18) that

$$-\ln\Delta_g(\zeta^*,\lambda^{(2)}) \geqq -\ln\varepsilon. \tag{3.3.20}$$

On the other hand, it follows from (3.3.19), since the function is analytic in the polycylinder $D'\times D''$, that

$$-\ln\Delta_g(\zeta,\lambda^{(2)}) \leqq -\ln 2\varepsilon \qquad \forall\,\zeta\in D',$$

and hence, by Lemma 3.3.3,

$$-\ln\Delta_g(\zeta^*,\lambda^{(2)}) = \lim_{\substack{\zeta\to\zeta^*\\ \zeta\in D'}}(-\ln\Delta_g(\zeta,\lambda^{(2)})) \leqq -\ln 2 - \ln\varepsilon.$$

But this contradicts (3.3.20). Consequently there can be no points of holomorphy

[24] This is a well-known result of the theory of analytic functions of several variables (see, for example, Fuks [2], Vladimirov [1], Šabat [1] or Hörmander [2]). The proof is given here for considerations of completeness.

for $g(\zeta, \lambda)$ on the manifold $\{(\zeta, \lambda): \zeta = \zeta^*, \lambda \in D''\}$, and the proof is complete.

We now consider an entire function $f(z)$ of finite order. Let $\rho_1 > 0, \cdots, \rho_n > 0$ be a system of associated orders. We can write the power-series expansion of $f(z)$ in the form

$$f(z) = \sum_{\|k\|=0}^{\infty} \frac{a_k z^k}{\Gamma(1 + k_1/\rho_1) \cdots \Gamma(1 + k_n/\rho_n)}.$$

Define the associated function of $f(z)$ to be

$$F_{f,\rho}(z) = \sum_{\|k\|=0}^{\infty} \frac{a_k}{z^{k+1}}.$$

Now that we have proved Theorem 3.3.6, the analog of Pólya's theorem for this case is proved in the same way as the corresponding theorem in § 3.1, and is therefore presented below without proof. For the same reason, the analogs of Theorem 3.3.3 and the representations (3.3.4) and (3.3.5) are also presented without proofs.

The following theorem is the analog of Borel's theorem for entire functions of arbitrary order.

THEOREM 3.3.7. *Let S_σ be the hypersurface of associated types of an entire function $f(z)$ for associated orders $\rho_1 > 0, \cdots, \rho_n > 0$. Then the hypersurface whose points are $(\sigma_1^{1/\rho_1}, \cdots, \sigma_n^{1/\rho_n})$, where $(\sigma_1, \cdots, \sigma_n) \in S_\sigma$, coincides with the hypersurface of associated radii of convergence of the series defining the associated function $F_{f,\rho}(z)$.*

As in the case of functions of order 1, the following integral representations of the functions $f(z)$ and $F_{f,\rho}(z)$ are valid:

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{l_1} \cdots \int_{l_n} F_{f,\rho}(\zeta) \left\{\prod_{j=1}^{n} E_{\rho_j}(z_j\zeta_j)\right\} d\zeta_1 \wedge \cdots \wedge d\zeta_n, \tag{3.3.21}$$

$$F_{f,\rho}(z) = \prod_{j=1}^{n} \left\{\frac{\rho_j}{z_j} (e^{-i\varphi_j} z_j)^{\rho_j}\right\} \int_0^{\infty} \cdots \int_0^{\infty} f(t_1 e^{-i\varphi_1}, \cdots, t_n e^{-i\varphi_n}) \times \exp\left\{-\sum_{j=1}^{n} t_j^{\rho_j} (e^{-i\varphi_j} z_j)^{\rho_j}\right\} dt_1 \cdots dt_n, \tag{3.3.22}$$

where

$$E_{\rho_j}(\zeta_j) = \sum_{m=0}^{\infty} \frac{\zeta_j^m}{\Gamma(1 + m/\rho_j)},$$ [25]

and l_j are closed contours such that the function $F_{f,\rho}(z)$ is analytic [26] whenever

[25] For the properties of the function $E_\rho(\zeta)$, which is known as the Mittag-Leffler function, see, for example, Džrbašjan [1].

[26] As in § 3.1, $F_{f,\rho}(z)$ denotes not only the function defined as the sum of this series but also its analytic continuation from a neighborhood of the point at infinity to suitable domains in C^n.

each coordinate z_j is either on l_j or outside it. Of these representations, (3.3.21) is valid throughout C^n, while (3.3.22) is valid in the domain

$$G_{\nu,\varphi} = \{z: \operatorname{Re}(z_j e^{-i\varphi_j})^{\rho_j} > \nu_j, \left|\arg z_j - \varphi_j\right| < \pi/2\rho_j, j = 1, \cdots, n\},$$

where the numbers $\nu_1 \geqq 0, \cdots, \nu_n \geqq 0$ are so chosen that for some constant $C_{\nu,\varphi}$ the inequality

$$\ln \left|f(t_1 e^{i\varphi_1}, \cdots, t_n e^{i\varphi_n})\right| \leqq C_{\nu,\varphi} + \nu_1 t_1^{\rho_1} + \cdots + \nu_n t_n^{\rho_n}$$

holds for all $t_1 \geqq 0, \cdots, t_n \geqq 0$. The set of all such points $\nu = (\nu_1, \cdots, \nu_n)$ will be denoted by $T_{f,\rho}(\varphi)$. We also denote by $C_{f,\rho}(\varphi)$ the set of all points $\nu \in R_+^n$ such that the function $F_{f,\rho}(z)$ is analytic in the corresponding domain $G_{\nu,\varphi}$. The following theorem is the analog of Pólya's theorem for functions of several variables.

THEOREM 3.3.8 (RONKIN [3]).[27] *Let $f(z)$ be an entire function of at most normal type for associated orders $\rho_1 > 0, \cdots, \rho_n > 0$. Then for any $\varphi \in R^n$ the set $C_{f,\rho}(-\varphi)$ coincides with the closure of the set $T_{f,\rho}(\varphi)$.*

Without going into the details of the proof of this theorem, which as we have stated is essentially analogous to that of Theorem 3.3.4, we cite the following lemma, which must be used in the proof.

LEMMA 3.3.4. *The representation* (3.3.21) *is valid if for $\nu \in C_{f,\rho}(\varphi)$ the contours l_j are the closed contours $l_j = l_j(R, \varepsilon, \varphi)$ formed for given $\varepsilon > 0$ by an arc of the circle $|\zeta_j| = R$ for sufficiently small R and an arc of the curve*

$$\{z_j: \operatorname{Re}(z_j e^{-i\varphi_j})^{\rho_j} = \nu_j + \varepsilon, \left|\arg z_j - \varphi_j\right| \leqq \pi/2\rho_j\}.$$

Moreover, as in the case $\rho_f = 1$, *the sets $C_{f,\rho}(\varphi)$ and $\bar{T}_{f,\rho}(\varphi)$ are convex.*

§ 4. Fourier transform of entire functions of exponential type

We now consider entire functions of exponential type which are L_2 functions, for real values of $z_1, \cdots, z_n$, i.e. entire functions $f(z)$ such that for some $a > 0$ and $A > 0$ we have $|f(z)| \leqq Ae^{a|z|}$ for all $z \in C^n$, and $\int_{R^n} |f(x)|^2\, dx < \infty$. The growth characteristics of such functions are closely related to the properties of their Fourier transforms.

1. *Estimate of the L_p norm.* Any restrictions on the growth of an entire function of exponential type for real values of $z_1, \cdots, z_n$ usually entail certain restrictions on its growth as a function of $x_1 = \operatorname{Re} z_1, \cdots, x_n = \operatorname{Re} z_n$ for fixed values of $y_1 = \operatorname{Im} z_1, \cdots, y_n = \operatorname{Im} z_n$. We have already observed a similar phenomenon in the preceding section (see Theorem 3.3.5). The following lemma is a further illustration.

[27] A corresponding generalization of Pólya's theorem for entire functions of one variable was obtained by Subbotin [1]. See also Lohin [1] and Avetisjan [1]. Geče [2] has extended Theorem 3.3.8 to the case of entire functions with associated proximate orders.

LEMMA 3.4.1. *Let $f(z)$ be an entire function such that, for some positive constants $\sigma_1, \cdots, \sigma_n, b_1, \cdots, b_n$ and C, for all $z \in C^n$,*

$$|f(z)| \leqq C \exp\left\{\sum_{j=1}^{n} b_j |x_j| + \sum_{j=1}^{n} \sigma_j |y_j|\right\}. \tag{3.4.1}$$

Moreover, let

$$\|f(x)\|_p = \left(\int_{R^n} |f(x)|^p \, dx\right)^{1/p} < \infty \qquad (p > 0).$$

Then for any $y \in R^n$

$$\left(\int_{R^n} |f(x + iy)|^p \, dx\right)^{1/p} \leqq \|f(x)\|_p \exp\left\{\sum_{j=1}^{n} \sigma_j |y_j|\right\}.$$

PROOF. Consider the function

$$\alpha_M(z) = \int_{-M}^{M} \cdots \int_{-M}^{M} |f(z_1 + t_1, \cdots, z_n + t_n)|^p \, dt_1 \cdots dt_n.$$

In § 1 of Chapter 2 we saw that integration with respect to a parameter preserves the property of logarithmic plurisubharmonicity. The function $\alpha_M(z)$ is therefore logarithmically plurisubharmonic in C^n. Let us estimate it. For any $x \in R^n$ and any $M > 0$ we have

$$\alpha_M(x) \leqq \int_{R^n_{(t)}} |f(x + t)|^p \, dt = \|f(x)\|_p^p. \tag{3.4.2}$$

Further, using (3.4.1) we conclude that

$$\begin{aligned} \alpha_M(z) &\leqq C^p (2M)^n \exp\left\{p \sum_{j=1}^{n} \sigma_j |y_j| + p \sum_{j=1}^{n} b_j |x_j| + p \sum_{j=1}^{n} M b_j\right\} \\ &= \text{const} \cdot \exp\left\{p\left(\sum_{j=1}^{n} \sigma_j |y_j| + \sum_{j=1}^{n} b_j |x_j|\right)\right\}. \end{aligned} \tag{3.4.3}$$

Since the function $\alpha_M(z)$ is logarithmically plurisubharmonic, it is logarithmically subharmonic in each of the variables $z_1, \cdots, z_n$. We now apply the Phragmén-Lindelöf principle for logarithmically subharmonic functions in a half-plane, with respect to each variable $z_1, \cdots, z_n$ in turn. That this is possible follows from estimates (3.4.2) and (3.4.3). For any $M > 0$,

$$\begin{aligned} \alpha_M(z_1, x_2, \cdots, x_n) &\leqq \|f(x)\|_p^p \exp(p \sigma_1 |y_1|), \\ \alpha_M(z_1, z_2, x_3, \cdots, x_n) &\leqq \|f(x)\|_p^p \exp(p(\sigma_1 |y_1| + \sigma_2 |y_2|)), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_M(z_1, \cdots, z_n) &\leqq \|f(x)\|_p^p \exp(p \textstyle\sum_1^n \sigma_j |y_j|)). \end{aligned} \tag{3.4.4}$$

The last inequality implies that

$$\int_{R^n_{(x)}} |f(z+t)|^p \, dt \leqq \|f(x)\|_p^p \exp(p \textstyle\sum_1^n \sigma_j |y_j|).$$

Setting $z = iy$, $y \in R^n$, in this estimate, we get

$$\int_{R^n_{(x)}} |f(t+iy)|^p \, dt \leqq \|f(x)\|_p^p \exp(p \textstyle\sum_1^n \sigma_j |y_j|),$$

and thus

$$\left(\int_{R^n} |f(x+iy)|^p \, dx\right)^{1/p} \leqq \|f(x)\|_p \exp(\textstyle\sum_1^n \sigma_j |y_j|).$$

Q.E.D.

COROLLARY. *Any entire function $f(z)$ satisfying the assumptions of Lemma* 3.4.1 *is bounded for real values of $z_1, \cdots, z_n$.*

Indeed, since the function $|f(z)|^p$ is subharmonic, it follows that for any $x_0 \in R^n$

$$\begin{aligned}
|f(x_0)| &\leqq \left(\frac{1}{\omega_{2n}} \int_{|z-x_0|<1} |f(z)|^p \, dz\right)^{1/p} \leqq \left(\frac{1}{\omega_{2n}} \int_{\substack{|\operatorname{Im} z_j|<1 \\ j=1,\cdots,n}} |f(z)|^p \, dz\right)^{1/p} \\
&= \left\{\frac{1}{\omega_{2n}} \int_{-1}^{1} \cdots \int_{-1}^{1} \left\{\int_{R^n_{(x)}} |f(x+iy)|^p \, dx\right\} dy\right\}^{1/p} \\
&\leqq \left\{\frac{\|f(x)\|_p^p}{\omega_{2n}} \int_{-1}^{1} \cdots \int_{-1}^{1} \exp(p \textstyle\sum \sigma_j |y_j|) \, dy_1 \cdots dy_n\right\}^{1/p} \\
&= \left\{\frac{2^n}{\omega_{2n}}\right\}^{1/p} e^{\sum_1^n \sigma_j} \|f(x)\|_p.
\end{aligned}$$

Moreover, the procedure that led to inequality (3.4.4), namely successive application of the Phragmén-Lindelöf principle for a half-plane to an entire function of exponential type in each variable, yields the following proposition, which resembles Lemma 3.4.1.

LEMMA 3.4.2. *Let $f(z)$ be an entire function which has upper type $\bar{\sigma}_1 < \infty, \cdots, \bar{\sigma}_n < \infty$, respectively, in each of the variables $z_1, \cdots, z_n$, for order* 1, *and let $|f(x)| \leqq M$ for all $x \in R^n$. Then*

$$|f(x+iy)| \leqq M \exp(\textstyle\sum \bar{\sigma}_j |y_j|).$$

2. *Generalization of the Paley-Wiener Theorem.* The relationship of the growth of an entire function of one variable with the properties of its Fourier transform is embodied in the well-known Paley-Wiener Theorem (see, for example, Levin [1]):

THEOREM 3.4.1. *A function $f(z)$, $z \in C^1$, is an entire function of exponential type, in L_2 on the real axis, if and only if it admits the representation*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_a^b \Phi(t)\, e^{-itz}\, dt,$$

where $\Phi(t) \in L_2$. *When this representation is valid, the smallest closed interval* $[a, b]$ *containing the support of the function* $\Phi(t)$ *is given by*

$$a = -\overline{\lim_{y\to+\infty}} \frac{1}{y} \ln|f(-iy)|, \qquad b = \overline{\lim_{y\to+\infty}} \frac{1}{y} \ln|f(iy)|.$$

In order to formulate the analogous theorem for $n > 1$, we follow Plancherel and Pólya [1], introducing a characteristic for the growth of the function $f(z)$ in different directions in the space $I^n = R^n_{(y)}$ of the variables $y_1 = \operatorname{Im} z_1, \cdots, y_n = \operatorname{Im} z_n$. Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a point of the unit sphere in R^n, defining the direction in question. Set

$$h_f(\lambda; x) = \overline{\lim_{R\to+\infty}} \frac{1}{R} \ln|f(x_1 + i\lambda_1 R, \cdots, x_n + i\lambda_n R)|$$

and

$$h_f(\lambda) = \sup_{x\in R^n} h_f(\lambda; x).$$

We call $h_f(\lambda)$ the P-indicator of the function $f(z)$. The following theorem extends the Paley-Wiener Theorem to the case of several variables.

THEOREM 3.4.2 (PLANCHEREL AND PÓLYA [1]). *A function* $f(z)$, $z \in C^n$, *is an entire function of exponential type, in* L_2 *on* $R^n_{(x)}$, *if and only if it admits the representation*

$$f(z) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{R^n_{(t)}} \Phi(t) e^{-\langle t,z\rangle i}\, dt \tag{3.4.5}$$

where $\Phi(t) \in L_2$ *vanishes identically off some bounded domain. When this representation is valid, the support function* $H_{D_\Phi}(\lambda)$ *of the smallest convex domain* D_Φ *off which* $\Phi(t) \equiv 0$ *coincides for any* λ *with the P-indicator* $h_f(\lambda)$ *of the function* $f(z)$.

PROOF. (1) SUFFICIENCY. Let $\Phi(t) \in L_2$ and $\Phi(t) \equiv 0$ off a bounded convex domain D. Then the function $\Phi(t)$ is also in L_1 on R^n, and therefore the following integral is meaningful:

$$\int_{R^n_{(t)}} \Phi(t)\, e^{-\langle t,z\rangle i}\, dt = \int_D \Phi(t)\, e^{-\langle z,t\rangle i}\, dt.$$

Suppose that

$$f(z) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_D \Phi(t)\, e^{-\langle t,z\rangle i}\, dt. \tag{3.4.6}$$

We claim that the function $f(z)$ then has the required properties. Indeed, since D is bounded and $\Phi(t)$ integrable over D, it follows that the integral (3.4.6) is absolute-

ly and uniformly convergent in any bounded region of C^n. Consequently $f(z)$ is an entire function. Estimating this function, we get

$$\begin{aligned} |f(x + i\lambda R)| &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left| \int_D \Phi(t) \exp\{-\langle t, x\rangle i + \langle t, \lambda\rangle R\}\, dt \right| \\ &\leqq \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_D \left| |\Phi(t)|\, e^{\langle t,\lambda\rangle R}\, dt \right| \\ &\leqq \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{R \cdot \sup_{t\in D} \langle t, \lambda\rangle\right\} \int_D |\Phi(t)|\, dt = \text{const} \cdot e^{RH_D(\lambda)}. \quad (3.4.7) \end{aligned}$$

It follows from this inequality that for any $z \in C^n$

$$|f(z)| \leqq \text{const} \cdot e^{\alpha|y|} \leqq \text{const}\, e^{\alpha|z|},$$

where $\alpha = \sup_{|\lambda|=1} H_D(\lambda)$. Consequently $f(z)$ is a function of exponential type. This proves sufficiency.

We now estimate the P-indicator of $f(z)$. From (3.4.7) we get $h_f(\lambda; x) \leqq H_D(\lambda)$, and so

$$h_f(\lambda) \leqq H_D(\lambda). \quad (3.4.8)$$

Since the only condition imposed on the convex domain D is that $\Phi(t) \equiv 0$ outside D, inequality (3.4.8) is also valid for $D = D_\Phi$, i.e.

$$h_f(\lambda) \leqq H_{D_\Phi}(\lambda). \quad (3.4.9)$$

(2) NECESSITY. We first prove a lemma.

LEMMA 3.4.3. *Let $f(z)$, $z \in C^1$, be an entire function of exponential type satisfying the following conditions*:

(a) $\overline{\lim}_{y\to+\infty}\, y^{-1} \ln |f(iy)| = \sigma$.

(b) $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

Then the Fourier transform

$$\tilde{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{itx}\, dx$$

vanishes for all $t > \sigma$.

PROOF. Let l_R be the contour formed by the closed interval $I_R = \{x: |x| \leqq R\}$ on the real axis and the semicircle $C_R = \{z: |z| = R, \operatorname{Im} z > 0\}$. It is obvious that

$$\int_{l_R} f(z) \frac{\sin \varepsilon z}{\varepsilon z}\, e^{itz}\, dz = 0. \quad (3.4.10)$$

Now, by the Corollary to Lemma 3.4.1 and by Lemma 3.4.2,

$$\sup_{\operatorname{Im} z \geqq 0} |f(z)\, e^{i(\sigma+\varepsilon)z} \sin \varepsilon z| < \infty.$$

Hence by the Jordan lemma it follows that for $t > \sigma + \varepsilon$

$$\lim_{R\to\infty} \int_{C_R} f(z) \frac{\sin \varepsilon z}{\varepsilon z} e^{itz}\, dz = 0.$$

Consequently for $t > \sigma + \varepsilon$

$$\int_{-\infty}^{\infty} f(x) \frac{\sin \varepsilon x}{\varepsilon x} e^{itx}\, dx = 0.$$

Now we know that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \varepsilon x}{\varepsilon x} e^{itx}\, dx = \begin{cases} 0 & \text{for } |t| > \varepsilon, \\ 1/2\varepsilon & \text{for } |t| < \varepsilon, \end{cases}$$

and moreover the Fourier transform of a product is the convolution of the Fourier transforms of the factors. Thus

$$\frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \bar{f}(s)\, ds = 0 \qquad \forall t > \sigma + \varepsilon.$$

Letting $\varepsilon \to 0$, we see that $\bar{f}(t) = 0$ for all t in the interval (σ, ∞). Q.E.D.

Now let $f(z)$ be an entire function of exponential type in C^n, in L_2 on $R^n_{(x)}$. It follows from the theory of the Fourier transform in L_2[28] that $f(x)$ admits the representation

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \text{l.i.m.} \int_{R^n_{(t)}} \bar{f}(t)\, e^{-\langle t,x\rangle i}\, dt, \tag{3.4.11}$$

where

$$\bar{f}(t) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \text{l.i.m.} \int_{R^n_{(x)}} f(x)\, e^{\langle t,x\rangle i}\, dx. \tag{3.4.12}$$

As usual, the symbol l.i.m. denotes convergence of the integral in the L_2 metric. Thus (3.4.12) is equivalent to

$$\lim_{\substack{M_i\to\infty \\ M_i'\to\infty \\ i=1,\cdots,n}} \int_{R^n_{(t)}} \left| f(t) - \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{-M_1}^{M_1'} \cdots \int_{-M_n}^{M_n'} f(x) e^{\langle t,x\rangle i}\, dx_1, \cdots, dx_n \right|^2 dt = 0.$$

Now assume that in addition $f(x) \in L_1$. Then the Fourier transform $\bar{f}(t)$ of $f(x)$ exists not only in the l.i.m. sense but also in the ordinary sense:

$$\bar{f}(t) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{R^n_{(x)}} f(x)\, e^{\langle t,x\rangle i}\, dx.$$

Consider a straight line through the origin in $R^n_{(x)}$, and make this line the x_1'-axis in

[28] See, for example, Bochner [1].

a new coordinate system $x_1', \cdots, x_n'$. Then $x_p = \sum_{q=1}^n x_q' \lambda_{p,q}$, where $\lambda_{p,q}$ is the cosine of the angle between the x_q' and x_p axes. This transformation of variables maps $f(x)$ onto the function

$$g(x') = f(a_1 + \lambda_{1,1}x_1', \cdots, a_n + \lambda_{n,1}x_n'),$$

where

$$a_p = \sum_{q=2}^n x_q' \lambda_{p,q} \qquad (p = 1, \cdots, n).$$

It is clear that, together with $f(x)$, the function $g(x')$ is also in L_1. Expressing $\int |g(x')|\,dx'$ as a multiple integral and using Fubini's Theorem, we see that $\int_{-\infty}^{\infty} |g(x')|\,dx_1'$ exists for almost all $(x_2', \cdots, x_n') \in R^{n-1}$. Applying the same transformation to the integral (3.4.12) and carrying out the integration with respect to x_1', we get

$$\tilde{f}(t) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{R^n_{(x')}} g(x') \exp\left\{i \sum_{p=1}^n t_p \sum_{q=1}^n x_q' \lambda_{p,q}\right\} dx'$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{R^{n-1}} e^{\sum_1^n t_p a_p}\, dx_2' \cdots dx_n' \int_{-\infty}^{\infty} g(x') \exp\left\{ix_1' \sum_{p=1}^n t_p \lambda_{p,1}\right\} dx_1'. \quad (3.4.13)$$

The inner integral in this expression exists for almost all $(x_2', \cdots, x_n')$. Moreover,

$$g(x') = f(a_1 + z_1' \lambda_{1,1}, \cdots, a_n + z_1' \lambda_{n,1})\big|_{y_1'=0},$$

where $z_1' = x_1' + iy_1'$. Consequently $g(x')$, as a function of x_1', can be continued to the complex z_1'-plane as an entire function of exponential type. Note also that

$$\overline{\lim_{R\to\infty}} \frac{1}{R} \ln |f(a_1 + iR\lambda_{1,1}, \cdots, a_n + iR\lambda_{n,1})|$$

$$= h_f(\lambda_{1,1}, \cdots, \lambda_{n,1}, a_1, \cdots, a_n) \leqq h_f(\lambda_{1,1}, \cdots, \lambda_{n,1}).$$

Hence, by Lemma 3.4.3, we conclude that for almost all $(x_2', \cdots, x_n')$, if $\sum_1^n t_p \lambda_{p,1} \geqq h_f(\lambda_{1,1}, \cdots, \lambda_{n,1})$, then

$$\int_{-\infty}^{\infty} g(x') \exp\left\{i \sum_{p=1}^n t_p \lambda_{p,1}\right\} dx_1' = 0.$$

In view of equality (3.4.13), we see moreover that $\tilde{f}(t) = 0$ in the half-space $\sum_1^n t_p \lambda_{p,1} \geqq h_f(\lambda_{1,1}, \cdots, \lambda_{n,1})$. Since the x_1'-axis can be chosen arbitrarily, it follows that $\tilde{f}(t) = 0$ outside the convex domain D which is the intersection of all the half-spaces $\langle t, \lambda\rangle \leqq h_f(\lambda)$ $(\lambda \in R^n)$. Consequently the support function H_{D_f} of the smallest convex domain $D_{\tilde{f}}$ outside which $\tilde{f}(t) = 0$ satisfies the condition

$$H_{D_f}(\lambda) \leqq h_{\tilde{f}}(\lambda).$$

Now consider the case in which $\tilde{f}(x)$ is in L_2 but not necessarily in L_1. Set

$$f_\varepsilon(z) = f(z) \prod_{j=1}^n \frac{\sin \varepsilon z_j}{\varepsilon z_j}.$$

We have $f_\varepsilon(x) \in L_1$, and as proved above the Fourier transform $\tilde{f}_\varepsilon(t)$ of this function vanishes outside some convex domain D_ε whose support function $H_{D_\varepsilon}(\lambda)$ satisfies the condition $H_{D_\varepsilon}(\lambda) \leqq h_f(\lambda)$ for any λ. Now,

$$\tilde{f}_\varepsilon(t) = \left(\frac{1}{2\varepsilon}\right)^n \int_{t_1-\varepsilon}^{t_1+\varepsilon} \cdots \int_{t_n-\varepsilon}^{t_n+\varepsilon} \tilde{f}(s)\, ds_1 \cdots ds_n,$$

and thus, for almost all $t \in R^n$,

$$\overline{\lim_{\varepsilon \to 0}} \tilde{f}_\varepsilon(t) = \tilde{f}(t).$$

Thus for every $\varepsilon > 0$ we have $\tilde{f}(t) = 0$ almost everywhere outside the convex domain whose support function is $h_f(\lambda) + \varepsilon$. Consequently the support function $H_{D_f}(\lambda)$ of the smallest convex domain $D_{\tilde{f}}$ outside which $\tilde{f}(t) = 0$ almost everywhere satisfies the condition

$$H_{D_f}(\lambda) \leqq h_f(\lambda). \tag{3.4.14}$$

Since $f(z)$ is of exponential type, its P-indicator $h_f(\lambda)$, and hence also the function $H_{D_f}(\lambda)$, are bounded. Consequently the domain $D_{\tilde{f}}$ is bounded. Hence by (3.4.11) it follows that the representation (3.4.5) is valid with $\Phi(t) = \tilde{f}(t)$. This proves necessity.

To complete the proof, it now suffices to observe that inequalities (3.4.14) and (3.4.9) imply $H_{D_\Phi}(\lambda) = h_f(\lambda)$ for all $\lambda \in R^n$. Q.E.D.

3. *Lowering of the indicator* $h_f(\lambda; x)$. We have repeatedly encountered the situation in which an entire function has the same growth (e.g., order, type, etc.) on almost all parallel planes or straight lines. One result in this category is the following theorem.

THEOREM 3.4.3 (RONKIN [2]). *Let $f(z)$, $z \in C^n$, be an entire function of exponential type. Then the set Q_f of points $x \in R^n$ at which (in the notation of* § 4.2) $h_f(\lambda; x) < h_f(\lambda)$ *has Lebesgue measure zero in R^n.*

PROOF. Consider the function

$$g_\lambda(z, w) = f(z_1 + \lambda_1 w, \cdots z_n + \lambda_n w) \qquad (w \in C^1).$$

It is obvious that

$$h_f(\lambda; x) = h_{g_{\lambda,w}}(\pi/2; x), \tag{3.4.15}$$

where $h_{g_{\lambda,w}}(\theta; z)$ denotes the ordinary indicator of the function $g_\lambda(z, w)$ in the variable w for fixed z. By Theorem 3.2.5 the inequality

$$h_{g_{\lambda,w}}(\pi/2; z) < h_{g_{\lambda,w}}(\pi/2),$$

where $\bar{h}_{g_{\lambda,w}}(\theta)$ is the upper indicator of the function $g_\lambda(z, w)$ in w, can hold only on a set N such that Γ-Cap $N = 0$. By Lemma 2.2.7 the intersection $N^* = N \cap R^n_{(x)}$ has Lebesgue measure zero in R^n. Therefore by (3.4.15) we have

$$\sup_{x \in R^n} h_{g_\lambda, w}(\pi/2; x) = \bar{h}_{g_\lambda, w}(\pi/2)$$

and consequently $\bar{h}_{g_\lambda}(\pi/2) = h_f(\lambda)$. Hence it follows that $Q_f = N^*$, and thus the set Q_f has Lebesgue measure zero in R^n. Q.E.D.

4. *Addition of P-indicators.* Let $f(z)$ and $\varphi(z)$ be entire functions of exponential type in C^n, and let $g(z) = f(z)\varphi(z)$. It is clear that the P-indicators of these functions satisfy the inequality

$$h_g(\lambda) \leqq h_f(\lambda) + h_\varphi(\lambda). \tag{3.4.16}$$

In this subsection we determine some sufficient conditions for equality to hold in (3.4.16). In so doing we shall make essential use of the class of entire functions of completely regular growth, a concept introduced by Levin and Pfluger. An entire function $f(w)$ of one complex variable w is said to have *completely regular growth* if for $\theta \in [0,2\pi]$ the limit

$$\lim_{R \to \infty} \frac{1}{R} \ln |f(Re^{i\theta})|$$

exists as R goes to infinity over all positive numbers except perhaps for some set E, independent of θ, such that

$$\lim_{r \to \infty} \frac{1}{r} \mu_1(E \cap (0, r)) = 0.^{29)}$$

In the theory of entire functions of one variable it is proved (see Levin [1]) that if at least one of the factors in the product $g(w) = f(w)\varphi(w)$ is a function of completely regular growth, then

$$\overline{\lim_{R \to \infty}} \frac{1}{R} \ln |g(Re^{i\theta})| = \overline{\lim_{R \to \infty}} \frac{1}{R} \ln |f(Re^{i\theta})| + \overline{\lim_{R \to \infty}} \frac{1}{R} \ln |\varphi(Re^{i\theta})|,$$

and hence, in this case, $h_g(\lambda) = h_f(\lambda) + h_\varphi(\lambda)$ $(\lambda = \pm 1)$. It is also known (Levin [1]) that a sufficient condition for an entire function of exponential type to have completely regular growth is the convergence of the integral

$$\int_{-\infty}^{\infty} \frac{\ln^+ |f(t)|}{1 + t^2} dt.$$

In particular, entire functions of exponential type which are bounded on the real axis are functions of completely regular growth.

THEOREM 3.4.4. *Let $\varphi(z)$ and $f(z)$ be entire functions of exponential type such that for almost all $x \in R^n$ the function $f(x + \lambda w)$, $\lambda \in R^n$, has completely regular growth in the variable $w \in C^1$. Then the P-indicator of the function $g(z) = \varphi(z)f(z)$ is the sum of the P-indicators of $f(z)$ and $\varphi(z)$: $h_g(\lambda) = h_f(\lambda) + h_\varphi(\lambda)$.*

[29)] Recall that $\mu_1(\cdot)$ denotes linear Lebesgue measure.

PROOF. By Theorem 3.4.3 the sets Q_f and Q_φ of points x such that $h_f(\lambda;x) < h_f(\lambda)$ and $h_\varphi(\lambda;x) < h_\varphi(\lambda)$ for some fixed λ have Lebesgue measure zero in R^n. On the other hand, by assumption the set of points x such that the function $f(x + \lambda w)$ is not a function of completely regular growth has Lebesgue measure zero. Consequently there exists a point $x' \in R^n_{(x)}$ such that

$$h_f(\lambda;x') = h_f(\lambda), \qquad h_\varphi(\lambda; x') = h_\varphi(\lambda), \tag{3.4.17}$$

and $f(x' + \lambda w)$ is a function of completely regular growth in w.

Consider the product $g(x' + \lambda w) = f(x' + \lambda w)\varphi(x' + \lambda w)$. By the above rule for addition of indicators of entire functions of one variable,

$$h_g(\lambda; x') = h_f(\lambda; x') + h_\varphi(\lambda; x').$$

Hence by (3.4.17) it follows that $h_g(\lambda) \geqq h_f(\lambda) + h_\varphi(\lambda)$. But, as we remarked previously, the converse inequality is always true. Thus $h_g(\lambda) = h_f(\lambda) + h_\varphi(\lambda)$. Q.E.D.

REMARK. It is easy to see that Theorems 3.4.4 and 3.4.3 remain valid for a more general definition of the indicator:

$$h_{f,\rho}(\theta; \lambda) = \sup_x h_{f,\rho}(\theta; \lambda; x),$$

where

$$h_{f,\rho}(\theta; \lambda; x) = \overline{\lim_{R\to\infty}} R^{-\rho} \ln \left| f(x_1 + \lambda_1 Re^{i\theta}, \cdots, x_n + \lambda_n Re^{i\theta}) \right|,$$

and $\lambda \in C^n$ and $\rho > 0$.

We have confined our discussion here to the P-indicator alone, this being the most important indicator.

To end this section, we remark that Theorem 3.4.2 can be generalized, first, to the case of functions which are not necessarily entire but analytic in tube domains

$$T_K = \{z = x + iy: x \in R^n, y \in K\},$$

where K is a cone in $R^n_{(y)}$ with apex at the origin, and, second, to functions which, instead of belonging to L_2 on $R^n_{(x)}$, satisfy certain other less restrictive conditions. For generalizations of the first type, see, for example, Martin [1] and Bochner and Martin [1], and for the second see Vladimirov [2], Schwartz [1] or Gel'fand and Šilov [2].

The P-indicator can also be defined quite naturally for plurisubharmonic functions in tube domains. The properties of such functions, their P-indicators and some related questions have been studied by Vladmirov [3], [4], [5] and Lelong [6], [7].

§ 5. Radial indicator

For entire functions of normal type for any order $\rho > 0$, and also entire func-

tions of exponential type which are unrestricted as to growth on $R^n_{(x)}$, a natural substitute for the P-indicator is

$$L_r(z) = L_r(z,f) = \overline{\lim_{t\to\infty}} \frac{1}{t^\rho} \ln |f(tz)|, \qquad z \in C^n,$$

and its regularization $L_r^*(z)$. Following Lelong [6], we shall call $L_r(z)$ the radial indicator and $L_r^*(z)$ the regularized radial indicator. For the sake of brevity we shall often omit the adjectives "radial" and "regularized." Some properties of the indicators $L_r(z)$ and $L_r^*(z)$ follow.

(a) *$L_r^*(z)$ is plurisubharmonic in C^n.*

Indeed, the family of plurisubharmonic functions $t^{-\rho} \ln |f(tz)|$, $0 < t < \infty$, is clearly uniformly bounded on any compact subset of C^n, and thus (a) is a special case of the general theorem on the upper limit of a family of plurisubharmonic functions.

(b) *$L_r^*(z)$ is a positively homogeneous function of degree ρ, i.e. $L_r^*(tz) - t^\rho L_r^*(z)$ for any $z \in C^n$ and $t > 0$.*

This is obvious.

(c) *If $L_r^*(z)$ is the indicator of an entire function $f(z)$ of order $\rho > 0$, then for every $z^0 \in C^n$, $|z^0| = 1$, and any $A > L_r^*(z)$ there exists a cone $V \subset C^n$, with apex at the origin, such that $z^0 \in V$ and for all $z \in V$, except perhaps points z in some bounded domain,*

$$\ln |f(z)| \leqq A|z|^\rho. \tag{3.5.1}$$

Indeed, since $L_r^*(z)$ is plurisubharmonic in C^n, it follows that for sufficiently small $\varepsilon > 0$ we have $L_r^*(z) \leqq A - \varepsilon$ in some neighborhood υ of z^0. Hence by Hartogs' Lemma it follows that in every neighborhood $\upsilon' \Subset \upsilon$, $\upsilon' \ni z^0$, for any $\varepsilon_1 > 0$, and for all t from some $t' = t'(\upsilon', \varepsilon_1)$ on,

$$t^{-\rho} \ln |f(tz)| \leqq A - \varepsilon + \varepsilon_1.$$

It follows immediately from this inequality that (3.5.1) holds for all z in the cone $V = \{z: z/|z| \in \upsilon'\}$ such that $|z| > t'$.

(d) *$L_r^*(z)$ is independent of the choice of the origin, i.e. for any $z^0 \in C^n$ the regularized radial indicators of the functions $f(z)$ and $f(z^0 + z)$ are identical.*

Indeed, any cone V with apex at the origin containing the point z' also contains all points $z = z^0 + tz'$ for sufficiently large t. By property (c) the radial indicator $L_r(z; z^0)$ of the function $f(z^0 + z)$ is related to the indicator $L_r^*(z) = L_r^*(z; 0)$ by the inequality $L_r(z; z^0) \leqq L_r^*(z)$. Consequently $L_r^*(z; z^0) \leqq L_r^*(z)$. The converse inequality is proved in analogous fashion. Hence for any $z^0 \in C^n$

$$L_r^*(z; z^0) = L_r^*(z).$$

(e) *Let $l(z)$ be a positively homogeneous continuous function of degree ρ such that $L_r^*(z) \leqq l(z)$. Then for any $\varepsilon > 0$ there exists a constant C_ε such that for all $z \in C^n$*

$$\ln|f(z)| \leqq C_\varepsilon + l(z) + \varepsilon|z|^\rho. \tag{3.5.2}$$

Indeed, it follows from (c) that for any fixed $\varepsilon > 0$ every point $z' \in \{z: |z| = 1\}$ has a neighborhood $\upsilon_{z'}$ on the unit sphere such that for $z/|z| \in \upsilon_{z'}$ and some constant $C(\varepsilon, z')$

$$\ln|f(z)| \leqq (l(z/|z|) + \varepsilon)|z|^\rho + C(\varepsilon, z').$$

It is clear that each neighborhood $\upsilon_{z'}$ may be taken so small that $l(z) > l(z') - \varepsilon$ for $z \in \upsilon_{z'}$. Let $\{\upsilon_{z'_{(j)}}\}_{j=1}^N$ be a finite family of these neighborhoods $\upsilon_{z'}$ covering the unit sphere. Then for any z the point $z/|z|$ is in one of these neighborhoods, and consequently

$$\begin{aligned}\ln|f(z)| &\leqq \left(l\left(\frac{z'_{(j)}}{|z'_j|}\right) + \varepsilon\right)|z|^\rho + \max_{1\leqq j\leqq N} C(\varepsilon, z'_{(j)}) \\ &\leqq \left(l\left(\frac{z}{|z|}\right) + 2\varepsilon\right)|z|^\rho + \max_{1\leqq j\leqq N} C(\varepsilon, z'_{(j)}) \\ &= l(z) + 2\varepsilon|z|^\rho + \max_{1\leqq j\leqq N} C(\varepsilon, z'_{(j)}).\end{aligned}$$

(f) *The set N of points z such that $L_r(z) < L_r^*(z)$ is an F_σ set, and Γ-Cap $N = 0$.*

This is an obvious consequence of Theorem 2.3.2.

Many properties of the indicator $L_r^*(z)$ already follow from its being plurisubharmonic and positively homogeneous. They are therefore stated below as properties of positively homogeneous plurisubharmonic functions.

(g) *If $u(z)$ is a positively homogeneous plurisubharmonic function in C^n of degree $\rho > 0$, then there exists a monotone decreasing sequence of positively homogeneous functions $u_j(z)$, of degree ρ, infinitely differentiable in $C^n\backslash\{0\}$ and plurisubharmonic in C^n, such that*

$$\lim_{j\to\infty} u_j(z) = u(z),$$

and for any function $g(z)$, continuous in C^n and majorizing the function $u(z)$, the sequence $\{\max[u_{j,g}]\}_1^\infty$ converges to $g(z)$ uniformly on every compact set $K \subset C^n$ (*Kiselman* [3]).

To prove this assertion, we consider in the space C^{n^2} of the complex variables $a_{p,q} = \xi_{p,q} + i\eta_{p,q}$ $(p, q = 1, \cdots, n)$ the functions

$$\Phi_m(a) = \prod_{p\neq q} \alpha(|a_{p,q}|) \cdot \prod_{p=1}^n \alpha(|a_{p,p} - m|), \qquad m = 1,2, \cdots,$$

where $\alpha(t)$ is some infinitely differentiable function vanishing for $t \notin (0,1)$ and non-negative on the interval (0, 1), such that

$$\int_0^1 t\alpha(t)\, dt = 1/2\pi.$$

It is clear that the functions $\Phi_m(a)$ are infinitely differentiable in the space R^{2n^2} of the variables $\xi_{p,q}$ and $\eta_{p,q}$, and $\int_{C^{n^2}} \Phi_m(a)da = 1$.

Denote $F(z, a) = u(\zeta)$, where $\zeta = (\zeta_1, \cdots, \zeta_n)$ and $\zeta_p = \sum_{q=1}^{n} a_{p,q} z_q$ for $p = 1, \cdots, n$, and consider the functions

$$u_m(z) = m^{2n^2} \int_{C^{n^2}} F(z, a) \Phi_m(ma)\, da = m^{2n^2} \int_{D_m} F(z, a) \Phi(ma)\, da, \tag{3.5.3}$$

where D_m is the polycylinder

$$D_m = \{a: |a_{p,q}| < 1/m \text{ for } p \neq q; |a_{p,p} - 1| < 1/m \text{ for } p = 1, \cdots, n\}.$$

For any fixed a the function $F(z, a)$ is positively homogeneous of degree ρ and plurisubharmonic in C^n. Thus $u_m(z)$ is also positively homogeneous and plurisubharmonic for any m. Further, for any fixed z the function $F(z, a)$ is a plurisubharmonic function of the variables $a_{p,q}$, and the functions $u_m(z, a) = u_m(\zeta)$ are constructed from the function $F(z, a)$ by the method of Chapter 2, § 1, for construction of a sequence of infinitely differentiable plurisubharmonic functions converging to a given function. Hence, without going into details, we may assert that the functions $u_m(z)$ form a monotone decreasing sequence such that

$$\lim_{m \to \infty} u_m(z) = F(z, a^0),$$

where $a^0_{p,q} = 0$ for $p \neq q$ and $a^0_{p,p} = 1$, $p = 1, \cdots, n$. It is clear that $F(z, a^0) = u(z)$. Consequently $\lim_{m \to \infty} u_m(z) = u(z)$. We claim that the functions $u_m(z)$ are infinitely differentiable in $C^n \backslash \{0\}$. To prove this, take some point $z^0 \neq 0$ and let q' be such that $z^0_{q'} \neq 0$. We now perform a substitution of variables in the integral (3.5.3), going from $a_{p,q'}$ to $\zeta_p = \sum a_{p,q} z_q$. The only part of the integrand depending on the variables $z_1, \cdots, z_n$ will be $\Phi_m(ma)$, which is infinitely differentiable in these variables at the point z^0, Consequently the function $u_m(z)$ is also infinitely differentiable at this point.

Now consider some compact set K and a continuous majorant $g(z)$ of the function $u(z)$. Since $g(z)$ is continuous, it follows from Hartogs' Lemma that for any $\varepsilon > 0$ and all j from some $j_0 = j_0(\varepsilon, K)$ on we have $u_j(z) \leqq g(z) + \varepsilon$ for all $z \in K$. On the other hand, $u_j(z) \geqq u(z)$ for all $z \in C^n$ and $j = 1,2, \cdots$. Consequently, for $z \in K$ and $j > j_0$,

$$g(z) + \varepsilon \geqq \max [u_j(z), g(z)] \geqq \max [u(z), g(z)] = g(z)$$

and thus the sequence $\{\max [u_j, g]\}_1^\infty$ converges uniformly to $g(z)$. This completes the proof of (g).

(h) *Any function $u(z)$ positively homogeneous and subharmonic in C^1 is continuous.*

To prove this, it will clearly suffice to show that in any domain $G_\theta = \{z: |\arg z - \theta| < \pi\}$ the function $\Phi_\theta(z) = u(z^{1/\rho})$, where ρ is the degree of homogeneity of $u(z)$, is convex. By (g) we need only prove the convexity of $\Phi_\theta(z)$ for infinitely differentiable functions $u(z)$. But in this case, by positive homogeneity and subharmonicity, we have

$$x\, \partial\Phi_\theta/\partial x + y\, \partial\Phi_\theta/\partial y = \Phi_\theta \tag{3.5.4}$$

and

$$\partial^2\Phi_\theta/\partial x^2 + \partial^2\Phi_\theta/\partial y^2 \geqq 0. \tag{3.5.5}$$

It follows from (3.5.4) that

$$x\partial^2\Phi_\theta/\partial x^2 + y\,\partial^2\Phi_\theta/\partial x\,\partial y = 0, \qquad y\,\partial^2\Phi_\theta/\partial y^2 + x\,\partial^2\Phi_\theta/\partial x\,\partial y = 0,$$

and so

$$\partial^2\Phi_\theta/\partial x^2 \cdot \partial^2\Phi_\theta/\partial y^2 = (\partial^2\Phi_\theta/\partial x\,\partial y)^2.$$

Hence by (3.5.5), using the fact that $\Delta|z| = 1/|z|$, we conclude that for any $\varepsilon > 0$ the function $\Phi_\theta(z) + \varepsilon|z|$ is convex in the domain G_θ. Consequently the function $\Phi_\theta(z)$ is also convex in G_θ, and this completes the proof.

(i) *If $u(z) \not\equiv -\infty$ is a positively homogeneous plurisubharmonic function in C^n of degree $\rho > 0$, then for any $z \in C^n$*

$$u(e^{i\pi/\rho} z) + u(z) \geqq 0. \tag{3.5.6}$$

We first prove this for $n = 1$. As in (g), we consider the function $\Phi_\theta(z) = u(z^{1/\rho})$ in the domain $G_\theta = \{z: |\arg z - \theta| < \pi\}$. As was shown in (g), this function is convex in G_θ, and since, as we proved above, the function $u(z)$ is continuous, the function $\Phi_\theta(z)$ is convex in the closed domain $G_\theta' = \{z: |\arg z - \theta| \leqq \pi/2\}$. Therefore, for any $z \in C^1$ and $\theta = \arg z + \pi/2$,

$$\Phi_\theta(z) + \Phi_\theta(-z) \geqq \Phi_\theta(0) = 0.$$

This immediately implies (3.5.6).

For the general case $n > 1$, we consider the function $\varphi_z(w) = u(wz)$, $z \in C^n$, $w \in C^1$. If $u(e^{i\pi/\rho} z) + u(z) > -\infty$, then $\varphi_z(w) \not\equiv -\infty$ and, as we proved above,

$$\varphi_z(e^{i\pi/\rho} w) + \varphi_z(w) \geqq 0 \qquad \forall w \in C^1.$$

Setting $w = 1$, we see that $u(e^{i\pi/\rho} z) + u(z) > 0$ whenever $u(e^{i\pi/\rho} z) + u(z) > -\infty$. Hence, since the points z' at which $u(e^{i\pi/\rho} z') + u(z') > -\infty$ are dense in C^n, it follows that for any $z \in C^n$

$$u(e^{i\pi/\rho} z) + u(z) = \varliminf_{z' \to z} (u(e^{i\pi/\rho} z') + u(z')) \geqq 0.$$

(j) *If the function $u(z)$ is positively homogeneous and plurisubharmonic in C^n, $u(z) \not\equiv -\infty$, then $\inf_K u(z) > -\infty$ on any compact set $K \subset C^n$.*

Indeed, otherwise, as follows from (i), we would have

$$\sup_{e^{i\pi/\rho} z \in K} u(e^{i\pi/\rho} z) = +\infty,$$

which contradicts the plurisubharmonicity of $u(z)$.

As already mentioned, properties (g) through (j) are also valid for the radial indicator $L_r^*(z)$, which is a positively homogeneous plurisubharmonic function. It is natural to ask whether any positively homogeneous plurisubharmonic function

is the radial indicator $L_r^*(z)$ for some entire function. For $n = 1$ and $\rho = 1$ the answer is positive, as follows from Pólya's theorem on the relation between the growth of an entire function and the distribution of singularities of its associated function. For $n = 1$ and arbitrary $\rho > 0$ the answer, again positive, follows from theorems on the relation between the growth of an entire function and the distribution of its zeros (see Levin [1]). For $n > 1$ the problem is far more difficult. A solution has been given by Kiselman [3] and Martineau [2], [3], though Kiselman considers only entire functions of the first order. The methods of Kiselman and Martineau are essentially different. Kiselman reduces the problem to the construction of an analytic function with prescribed distribution of singularities, while Martineau uses results of Hörmander concerning the continuation of an entire function from a subspace of C^n to the entire space, preserving certain estimates. We shall present Martineau's solution in the next section, after a presentation of Hörmander's results; the following theorem will be proved:

THEOREM 3.5.1. *For any positively homogeneous plurisubharmonic function $u(z)$ in C^n of degree $\rho > 0$ there exists an entire function such that $L_r^*(z; f) = u(z)$.*

Using this theorem, we shall now show that, in contrast to the case $n = 1$ (see property (i)), the indicator $L_r^*(z)$ of an arbitrary entire function $f(z)$, $z \in C^n$, $n > 1$, need not be continuous (see Lelong [8]). It will suffice to construct a positively homogeneous plurisubharmonic function $u(z_1, z_2)$ of degree 1 which is discontinuous at a point of C^2. We first consider the function

$$\Phi(z_1) = \sum_{j=2}^{\infty} \frac{1}{2^j} \ln \left| z_1 - \frac{1}{2} + \frac{1}{j} \right|.$$

This function is subharmonic in C^1, since its defining series converges uniformly on any compact set not containing the point $z_1 = 1/2$, and in the neighborhood of this point the partial sums of the series are monotone decreasing. Note that $\Phi(1/2) > -\infty$, and $\Phi(z_1) \leqq 2 \ln |z_1|/3$ for sufficiently large $|z_1|$. Now consider the function

$$u(z_1, z_2) = |z_2| e^{\Phi(z_1/z_2)}, \qquad z_2 \neq 0,$$
$$u(z_1, 0) = 0.$$

We claim that this function is plurisubharmonic in C^2. Indeed, at all points (z_1, z_2), $z_2 \neq 0$, it is obviously upper semicontinuous. The same holds at points $(z_1, 0)$, $z_1 \neq 0$, since

$$\overline{\lim_{z_2 \to 0}} u(z_1, z_2) \leqq \overline{\lim_{|z_2| \to 0}} |z_2| \exp\left(\frac{2}{3} \ln \left| \frac{z_1}{z_2} \right|\right) = 0.$$

Finally, upper semicontinuity of $u(z_1, z_2)$ at the point (0, 0) follows from the following considerations: for any $t > 0$,

$$\overline{\lim_{\substack{z_1\to 0,\ z_2\to 0\\ |z_1/z_2|\leqq t}}} u(z_1, z_2) = 0,$$

and for any sufficiently large $t > 0$,

$$\overline{\lim_{\substack{z_1\to 0,\ z_2\to 0\\ |z_1/z_2|\geqq t}}} u(z_1, z_2) = 0,$$

so that

$$\overline{\lim_{z_1\to 0,\ z_2\to 0}} u(z_1, z_2) = \max\left\{\overline{\lim_{\substack{z_1\to 0, z_2\to 0\\ |z_1/z_2|\leqq t}}} u(z_1, z_2), \overline{\lim_{\substack{z_1\to 0, z_2\to 0\\ |z_1/z_2|\leqq t}}} u(z_1, z_2)\right\} = 0.$$

It is obvious that $u(a_1 w + b_1, a_2 w + b_2)$ is subharmonic. Consequently $u(z_1, z_2)$ is plurisubharmonic. In addition, $u(z_1, z_2)$ is clearly positively homogeneous of degree 1, and discontinuous for $z_2 = 2z_1$. By Theorem 3.5.1 it is the radial indicator $L_r^*(z_1, z_2)$ of some entire function $f(z_1, z_2)$ of exponential type. This proves the existence of discontinuous indicators $L_r^*(z)$.

To end this section, we point out the relation between the radial indicator $L_r^*(z)$ of an entire function $f(z)$ of exponential type and its P-indicator $h_f(\lambda)$. As follows from the definition of the functions $L_r^*(z)$ and $h_f(z)$, and from property (d) of $L_r^*(z)$, the following inequality holds for any $\lambda \in R^n$, $|\lambda| = 1$:

$$L_r^*(i\lambda) \geqq h_f(\lambda) \geqq L_r(i\lambda).$$

It is not known whether there exists a function $f(z)$ such that for some λ we have the strict inequality

$$L_r^*(i\lambda) > h_f(\lambda). \tag{3.5.7}$$

In the special case of entire functions of exponential type in L_2 on R^n, inequality (3.5.7) is impossible, i.e.

$$L_r^*(i\lambda) = h_f(\lambda) \qquad \forall \lambda \in R^n,\ |\lambda| = 1. \tag{3.5.8}$$

This obviously follows from Theorem 3.4.2. Equality (3.5.8) can also be proved under weaker restrictions on the growth of $f(z)$ on $R^n_{(x)}$, for example, for functions of at most power growth on $R^n_{(x)}$. However, the general case remains open.

§ 6. Operator methods

In recent years, many significant results in the theory of entire functions have been accomplished by methods of functional analysis. These methods have been developed by Hörmander [1], Kohn [1], Morrey [1] and others. Our presentation essentially follows Hörmander [2],[30] and in § 6.4 Martineau [2], [3]. We assume that the reader is familiar with the basic concepts of functional analysis, and in particular of operator theory.

[30] We discuss only results having a direct bearing on the theory of entire functions.

1. *Preliminaries.* Let $\varphi(z)$ be a real-valued continuous function on an open set $\Omega \subset C^n$. $L^2(\Omega, \varphi)$ will denote the space of functions $f(z)$, Borel-measurable on Ω, for which the integral

$$\|f\|_\varphi^2 = \int_\Omega |f|^2 e^{-\varphi}\, d\omega_z \text{ [31]} \tag{3.6.1}$$

is finite. $L^2(\Omega, \varphi)$ is a Hilbert space, with inner product

$$(f, g)_\varphi = \int_\Omega f \bar{g} e^{-\varphi}\, d\omega_z. \tag{3.6.2}$$

It is obvious that any function in $L^2(\Omega, \text{loc})$, i.e. any function which is locally square-integrable on Ω, is in $L^2(\Omega, \varphi)$ for some function $\varphi(z)$. On the other hand, for any function $\varphi(z)$ we have $L^2(\Omega, \varphi) \subset L^2(\Omega, \text{loc})$. Let $L^2_{(p,q)}(\Omega, \varphi)$ denote the space of forms

$$\alpha = \sum_I{}' \sum_K{}' \alpha_{I,K}\, dz^I \wedge d\bar{z}^K \tag{3.6.3}$$

where

$$I = (i_1, \cdots, i_p), \qquad K = (k_1, \cdots, k_q), \qquad \alpha_{I,K} \in L^2(\Omega, \varphi),$$
$$dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \qquad d\bar{z}^K = d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}$$

and

$$\sum_K{}' = \sum_{1 \le k_1 < k_2 < \cdots < k_q \le n}.$$

It is clear from the definition that $L^2_{(0,0)}(\Omega, \varphi) = L^2(\Omega, \varphi)$.

Set

$$|\alpha|^2 = \sum_I{}' \sum_K{}' |\alpha_{I,K}|^2, \tag{3.6.4}$$

$$\|\alpha\|_\varphi^2 = \int_\Omega |\alpha|^2 e^{-\varphi}\, d\omega_z \tag{3.6.5}$$

and

$$(\alpha, \beta)_\varphi^{(p,q)} = \int_\Omega \Big(\sum_I{}' \sum_K{}' \alpha_{I,K} \bar{\beta}_{I,K}\Big) e^{-\varphi}\, d\omega_z. \tag{3.6.5}$$

$L^2_{(p,q)}(\Omega, \varphi)$, like $L^2(\Omega, \varphi)$, is a Hilbert space, with inner product as defined by (3.6.6). In analogy with $L^2_{(p,q)}(\Omega, \varphi)$ and $L^2(\Omega, \text{loc})$, we define the space $L^2_{(p,q)}(\Omega, \text{loc})$. By $D_{(p,q)}(\Omega)$ we denote the space of forms (3.6.3) with coefficients in the space $D(\Omega)$ of infinitely differentiable functions with compact support in Ω. For any function $\varphi(z)$, the space $D_{(p,q)}(\Omega)$ is dense in $L^2(\Omega, \varphi)$.

Consider a form $\alpha \in L^2_{(p,q)}(\Omega, \varphi)$. Denote

$$\bar{\partial}\alpha = \sum_I{}' \sum_K{}' \sum_{j=1}^n \frac{\partial \alpha_{I,K}}{\partial \bar{z}_j}\, d\bar{z}_j \wedge dz^I \wedge d\bar{z}^K. \tag{3.6.7}$$

[31] As before, $d\omega_z$ (or simply $d\omega$) denotes the volume element in $C^n_{(z)}$.

If the functions $\alpha_{I,k}$ are not differentiable, the derivatives $\partial\alpha_{I,k}/\partial\bar{z}_j$ in this expression are to be understood in the distribution-theoretic sense.

Now consider the spaces $L^2(C^n, \varphi)$ and $L^2_{(0,1)}(C^n, \varphi)$. Define an operator T from $L^2(C^n, \varphi)$ to $L^2_{(0,1)}(C^n, \varphi)$ by $Tf = \bar{\partial} f$. The domain of definition D_T of the operator T consists of all functions $f(z) \in L^2(C^n, \varphi)$ such that the form $\bar{\partial} f$ is in the space $L^2_{(0,1)}(C^n, \varphi)$. In other words, $f \in D_T$ if the derivatives $\bar{\partial} f/\partial\bar{z}_j$, as elements of $D^*(C^n)$, are regular functionals, and their defining functions belong to the space $L^2(C^n, \varphi)$. Similarly, using the operator $\bar{\partial}$ we define an operator S (i.e. $Sf = \bar{\partial} f$) from $L^2_{(0,1)}(C^n, \varphi)$ to $L^2_{(0,2)}(C^n, \varphi)$, and its domain of definition D_S. Since differentiation of distributions is a continuous operator, T and S are closed operators. Note, moreover, that the domains of definition D_T and D_S are dense in the spaces $L^2(C^n, \varphi)$ and $L^2_{(0,1)}(C^n, \varphi)$, respectively. This follows from the fact that the sets $D(C^n)$ and $D_{(0,1)}(C^n)$, which are subsets of D_T and D_S, respectively, are dense in these spaces.

The adjoint T^* of the operator T is defined, as usual, by

$$(\alpha, Tf)^{(0,1)}_{\varphi} = (T^*\alpha, f)_{\varphi} \qquad \forall f \in D_T. \tag{3.6.8}$$

Since T is a closed operator with domain dense in $L^2(C^n, \varphi)$, the operator T^* exists and its domain of definition D_{T^*} is dense in $L^2_{(0,1)}(C^n, \varphi)$. Let us determine the explicit form of the operator T^*. For $f \in D_T$ and $\alpha = \sum_1^n \alpha_j\, d\bar{z}_j$, $\alpha \in D_{T^*}$,

$$\begin{aligned}\int_{C^n} T^*\alpha \bar{f} e^{-\varphi}\, d\omega_z &= (T^*\alpha, f)_{\varphi} = (\alpha, Tf)^{(0,1)}_{\varphi} \\ &= \int_{C^n} \left\{\sum_{j=1}^n \alpha_j (\overline{Tf})_j\right\} e^{-\varphi}\, d\omega_z = \int_{C^n} e^{-\varphi} \sum_{j=1}^n \alpha_j \frac{\overline{\partial f}}{\partial z_j}\, d\omega_z \\ &= -\int_{C^n} \sum_{j=1}^n \frac{\partial(e^{-\varphi}\alpha_j)}{\partial z_j} \bar{f}\, d\omega_z.\end{aligned}$$

Consequently

$$T^*\alpha = -e^{\varphi} \sum_{j=1}^n \frac{\partial(e^{-\varphi}\alpha_j)}{\partial z_j}. \tag{3.6.9}$$

We now present some lemmas needed in the sequel.

LEMMA 3.6.1. *If $\varphi(z)$ is a differentiable function and the norm is defined by $\|\alpha\|^*_{\varphi} = \|\alpha\|_{\varphi} + \|T^*\alpha\|_{\varphi} + \|S\alpha\|_{\varphi}$, the set $D_{(0,1)}(C^n)$ is a dense subset of $D_{T^*} \cap D_S$.*

PROOF. Let $\eta(z)$ be a function infinitely differentiable in C^n and such that $0 \leqq \eta(z) \leqq 1$ for all $z \in C^n$, $\eta(z) = 0$ for $|z| > 1$, $\eta(z) = 1$ for $|z| < 1/2$. Denote $\eta_m(z) = \eta(z/m)$. It is clear that for all sufficiently large m

$$\sum_{j=1}^n \left|\frac{\partial \eta_m}{\partial \bar{z}_j}\right|^2 \leqq 1 \qquad \forall z \in C^n. \tag{3.6.10}$$

In view of the relation

$$S(\eta_m\alpha) - \eta_m S\alpha = \bar\partial\eta_m \wedge \alpha \qquad \forall\alpha \in D_S,^{32)} \tag{3.6.11}$$

it follows from (3.6.10) that for $\alpha \in D_S$

$$\begin{aligned}|S(\eta_m\alpha) - \eta_m S\alpha|^2 e^{-\varphi} &\leqq 2\sum_{i,j=1}^{n}\left|\frac{\partial\eta_m}{\partial\bar z_i}\right|^2 |\alpha_j|^2 e^{-\varphi} \\ &\leqq 2\sum_{j=1}^{n}|\alpha_j|^2 e^{-\varphi} = 2|\alpha|^2 e^{-\varphi}.\end{aligned} \tag{3.6.12}$$

Moreover,

$$\lim_{m\to\infty}|S(\eta_m\alpha) - \eta_m S\alpha|^2 = \lim_{m\to\infty}|\bar\partial\eta_m \wedge \alpha|^2 = 0. \tag{3.6.13}$$

By the dominated convergence theorem it follows from (3.6.12) and (3.6.13) that for $\alpha \in D_S$

$$\lim_{m\to\infty}\|S(\eta_m\alpha) - \eta_m S\alpha\|_\varphi = 0. \tag{3.6.14}$$

Now suppose that $\alpha \in D_{T^*}$. Then also $\eta_m\alpha \in D_{T^*}$. Indeed, for any function $u(z) \in D_T$ we have

$$\begin{aligned}(\eta_m\alpha, Tu)_\varphi^{(0,1)} &= (\alpha, \bar\eta_m Tu)_\varphi^{(0,1)} = (\alpha, T(\bar\eta_m u))_\varphi^{(0,1)} + (\alpha, \bar\eta_m Tu - T(\bar\eta_m u))_\varphi^{(0,1)} \\ &= (\eta_m T^*\alpha, u)_\varphi + (\alpha, \bar\eta_m Tu - T(\bar\eta_m u))_\varphi^{(0,1)} = (\eta_m T^*\alpha, u)_\varphi - (\alpha, u\bar\partial\eta_m)_\varphi^{(0,1)}.\end{aligned}$$

Consequently

$$|(\eta_m\alpha, Tu)_\varphi^{(0,1)}| \leqq (\|\eta_m T^*\alpha\|_\varphi + A\|\alpha\|_\varphi)\|u\|_\varphi \qquad (A = \max_{C^n}|\bar\eta_m|),$$

and thus the linear functional $(Tu, \eta_m\alpha)_\varphi^{(0,1)}$ on D_T is continuous. Thus $\eta_m\alpha \in D_{T^*}$.

We now estimate $\|T^*(\eta_m\alpha) - \eta_m T^*\alpha\|_\varphi$. For $\alpha \in D_{T^*}$ and $u \in D_T$ we have

$$\begin{aligned}|(T^*(\eta_m\alpha) - \eta_m T^*\alpha, u)_\varphi| &= |(\alpha, \bar\eta_m Tu - T(\bar\eta_m u))_\varphi| \\ &= |(\alpha, \bar\partial\bar\eta_m)_\varphi| \leqq \|\alpha\|_\varphi \cdot \|u\bar\partial\bar\eta_m\|_\varphi \leqq \|\alpha\|_\varphi\|u\|_\varphi \cdot \max_{C^n}|\partial\bar\eta_m|.\end{aligned} \tag{3.6.15}$$

Since D_T is dense in $L^2(C^n, \varphi)$, it follows from (3.6.15) that

$$\|T^*(\eta_m\alpha) - \eta_m T^*\alpha\|_\varphi \leqq \|\alpha\|_\varphi \cdot \max_{C^n}|\bar\partial\bar\eta_m|. \tag{3.6.16}$$

Since

$$\lim_{m\to\infty}\max_{C^n}|\bar\partial\bar\eta_m| = 0, \tag{3.6.17}$$

we conclude that

$$\lim_{m\to\infty}\|T^*(\eta_m\alpha) - \eta_m T^*\alpha\|_\varphi = 0 \qquad \forall\alpha \in D_{T^*}.$$

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32) It is easy to see that when $\alpha \in D_S$ we also have $\eta_m\alpha \in D_S$.

Hence by (3.6.14) it follows that $\eta_m \alpha \to \alpha$ as $m \to \infty$, in the norm $\|:\|_\varphi^*$, for any form $\alpha \in D_{T^*} \cap D_S$.

To complete the proof of the lemma, it is now sufficient to show that any form $\alpha \in D_{T^*} \cap D_S$ with compact support can be approximated with arbitrary accuracy in the norm $\|:\|_\varphi^*$ by elements of $D_{(0,1)}(C^n, \varphi)$. To prove this, consider some function $\nu(z) \in D(C^n)$ such that $\int_{C^n} \nu(z) d\omega_z = 1$. Denote

$$\nu_\varepsilon(z) = \varepsilon^{-2n} \nu\left(\frac{z}{\varepsilon}\right), \qquad \alpha * \nu_\varepsilon = \sum_{j=1}^{n} \alpha_j * \nu_\varepsilon \, d\bar{z}_j,$$

where $\alpha_j * \nu_\varepsilon$ is the ordinary convolution of the functions α_j and ν_ε. It is clear that the forms $\alpha * \nu_\varepsilon$ are in $D_{(0,1)}(C^n)$. Further, since for any $\varepsilon \in (0, 1)$ the support of the form $\alpha * \nu_\varepsilon$ is contained in a fixed compact set, it follows, as is easily seen, that $\|\alpha - \alpha * \nu_\varepsilon\|_\varphi \to 0$ as $\varepsilon \to 0$. Similarly, since $S(\alpha * \nu_\varepsilon) = S\alpha * \nu_\varepsilon$, we have $\|S\alpha - S(\alpha * \nu_\varepsilon)\|_\varphi \to 0$ as $\varepsilon \to 0$. Now consider the last term in the expression for the norm $\|:\|_\varphi^*$. It follows from (3.6.9) that

$$T^*(\alpha * \nu_\varepsilon) = (T^*\alpha) * \nu_\varepsilon - \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j} (\alpha_j * \nu_\varepsilon) + \left(\sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j} \alpha_j\right) * \nu_\varepsilon.$$

The expression on the right of this equality converges in the L^2-metric to $T^*\alpha$ as $\varepsilon \to 0$. Consequently it also converges to $T^*\alpha$ in the $L^2(C^n, \varphi)$-metric, and so

$$\lim_{\varepsilon \to 0} \|T^*(\alpha * \nu_\varepsilon) - T^*\alpha\|_\varphi = 0.$$

Thus $\lim_{\varepsilon \to 0} \|\alpha * \nu_\varepsilon - \alpha\|_\varphi^* = 0$. This completes the proof of the lemma.

LEMMA 3.6.2. *For every form $\alpha \in D_{(0,1)}(C^n)$ and any twice continuously differentiable function $\varphi(z)$,*

$$\sum_{i,j=1}^{n} \int_{C^n} \alpha_i \bar{\alpha}_j \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} e^{-\varphi} \, d\omega_z + \sum_{i,j=1}^{n} \int_{C^n} \left| \frac{\partial \alpha_i}{\partial \bar{z}_j} \right|^2 e^{-\varphi} \, d\omega_z = \|T^*\alpha\|_\varphi^2 + \|S\alpha\|_\varphi^2. \quad (3.6.18)$$

PROOF. Set

$$\delta_j f = \partial f / \partial z_j - f \, \partial \varphi / \partial z_j.$$

It is easily verified that

$$\int_{C^n} f_1 \frac{\overline{\partial f_2}}{\partial \bar{z}_j} e^{-\varphi} \, d\omega_z = - \int_{C^n} (\delta_j f_1) \bar{f}_2 \, e^{-\varphi} \, d\omega_z \qquad (f_1, f_2 \in D(C^n)) \quad (3.6.19)$$

and

$$\delta_j \frac{\partial f}{\partial \bar{z}_i} - \frac{\partial}{\partial \bar{z}_i} (\delta_j f) = f \frac{\partial^2 \varphi}{\partial \bar{z}_i \partial z_j} \qquad \forall f \in D(C^n). \quad (3.6.20)$$

It follows from these relations that

$$\int_{C^n} \delta_i\alpha_i\,\overline{\delta_j\alpha_j}\,e^{-\varphi}\,d\omega_z = -\int_{C^n} \alpha_i\,\overline{\frac{\partial}{\partial\bar z_i}(\delta_j\alpha_j)}\,e^{-\varphi}\,d\omega_z$$

$$= -\int_{C^n} \alpha_i\,\overline{\left(\delta_j\frac{\partial\alpha_j}{\partial\bar z_i} - \alpha_j\frac{\partial^2\varphi}{\partial z_j\partial\bar z_j}\right)}\,e^{-\varphi}\,d\omega_z$$

$$= \int_{C^n} \alpha_i\bar\alpha_j\frac{\partial^2\varphi}{\partial z_i\partial\bar z_j}\,e^{-\varphi}\,d\omega_z + \int_{C^n}\frac{\partial\alpha_i}{\partial\bar z_j}\cdot\overline{\frac{\partial\alpha_j}{\partial\bar z_i}}\,e^{-\varphi}\,d\omega_z.$$

Therefore

$$\int_{C^n}\sum_{i,j=1}^n\left((\delta_i\alpha_i)\,\overline{(\delta_j\alpha_j)} - \frac{\partial\alpha_i}{\partial\bar z_j}\,\overline{\frac{\partial\alpha_j}{\partial\bar z_i}}\right)e^{-\varphi}\,d\omega_z + \int_{C^n}\sum_{i,j=1}^n\left|\frac{\partial\alpha_i}{\partial\bar z_j}\right|^2 e^{-\varphi}\,d\omega_z$$
$$= \int_{C^n}\sum_{i,j=1}^n \alpha_i\bar\alpha_j\frac{\partial^2\varphi}{\partial z_i\partial\bar z_j}\,e^{-\varphi}\,d\omega_z + \int_{C^n}\sum_{i,j=1}^n\left|\frac{\partial\alpha_i}{\partial\bar z_j}\right|^2 e^{-\varphi}\,d\omega_z. \tag{3.6.21}$$

Now observe that by (3.6.9)

$$\int_{C^n}\sum_{i,j=1}^n (\delta_i\alpha_i)\,\overline{(\delta_j\alpha_j)}\,e^{-\varphi}\,d\omega_z = \int_{C^n}\left|\sum_{j=1}^n \delta_i\alpha_i\right|^2 e^{-\varphi}\,d\omega_z = \|T^*\alpha\|_\varphi^2. \tag{3.6.22}$$

Further, it also follows that

$$\|S\alpha\|_\varphi^2 = \|\bar\partial\alpha\|_\varphi^2 = \frac{1}{2}\int_{C^n}\sum_{i,j=1}^n\left|\frac{\partial\alpha_i}{\partial\bar z_j} - \frac{\partial\alpha_j}{\partial\bar z_i}\right|^2 e^{-\varphi}\,d\omega_z$$
$$= \frac{1}{2}\int_{C^n}\sum_{i,j=1}^n\left(\frac{\partial\alpha_i}{\partial\bar z_j} - \frac{\partial\alpha_j}{\partial\bar z_i}\right)\left(\overline{\frac{\partial\alpha_i}{\partial\bar z_j}} - \overline{\frac{\partial\alpha_j}{\partial\bar z_i}}\right)e^{-\varphi}\,d\omega_z \tag{3.6.23}$$
$$= \int_{C^n}\left\{\sum_{i,j=1}^n\left|\frac{\partial\alpha_i}{\partial\bar z_j}\right|^2 - \sum_{i,j=1}^n\frac{\partial\alpha_i}{\partial\bar z_j}\cdot\overline{\frac{\partial\alpha_j}{\partial\bar z_i}}\right\}e^{-\varphi}\,d\omega_z.$$

Formulas (3.6.21), (3.6.22) and (3.6.23) imply the required equality (3.6.18). Q.E.D.

LEMMA 3.6.3. *Let $\varphi(z)$ be a strictly plurisubharmonic function in C^n. Let $c(z)$ be a positive continuous function such that*

$$\sum_{i,j=1}^n\frac{\partial^2\varphi}{\partial z_i\partial\bar z_j}\,\zeta_i\bar\zeta_j \geqq c(z)\,|\zeta|^2 \qquad \forall\zeta\in C^n. \tag{3.6.24}$$

Then

$$\int_{C^n} c(z)|\alpha|^2\,e^{-\varphi}\,d\omega_z \leqq \|T^*\alpha\|_\varphi + \|S\alpha\|_\varphi \qquad \forall\alpha\in D_{T^*}\cap D_S. \tag{3.6.25}$$

PROOF. For $\alpha\in D_{(0,1)}(C^n)$, the statement of the lemma is an obvious consequence of Lemma 3.6.2. It is extended to the general case by Lemma 3.6.1. Moreover, since the function $\varphi(z)$ is strictly plurisubharmonic, the set of functions $c(z)$ satisfying (3.6.24) is not empty.

2. *Existence theorems.* We now prove a number of theorems on the existence of solutions to the equations $\bar{\partial} f = \alpha$ which satisfy certain growth estimates.

THEOREM 3.6.1. *Let $\varphi(z)$ be a strictly plurisubharmonic function, and $c(z)$ a positive continuous function satisfying inequality* (3.6.24). *Let the form $\alpha \in L^2_{(0,1)}(C^n, \varphi)$ be such that $\bar{\partial}\alpha = 0$ and*

$$\int_{C^n} |\alpha|^2 \frac{e^{-\varphi}}{c}\, d\omega < \infty. \tag{3.6.26}$$

Then there exists a function $f \in L^2(C^n, \varphi)$ which is a solution of the equation $Tf = \alpha$[33] *and satisfies the condition*

$$\int_{C^n} |f|^2 e^{-\varphi}\, d\omega \leqq \int_{C^n} |\alpha|^2 \frac{e^{-\varphi}}{c}\, d\omega. \tag{3.6.27}$$

PROOF. Let N_S be the set of all forms $\gamma \in D_S$ such that $S\gamma = 0$. If $\beta \in D_{T^*} \cap N_S$, it follows from (3.6.25) that

$$\int_{C^n} c|\beta|^2 e^{-\varphi}\, d\omega_z \leqq \|T^*\beta\|_\varphi.$$

Consequently for any $\beta \in D_{T^*} \cap N_S$ we have

$$|(\alpha, \beta)_\varphi^{(0,1)}| \leqq \|T^*\beta\|_\varphi \left(\int_{C^n} |\alpha|^2 \frac{e^{-\varphi}}{c}\, d\omega \right)^{1/2}. \tag{3.6.28}$$

Since N_S is a closed set, it follows that any form such that $\beta \in D_{T^*}$ but $\beta \notin N_S$ can be expressed as $\beta = \beta' + \beta''$, where $\beta' \in N_S$ and β'' is orthogonal to N_S. Since the range of the operator T is obviously a subset of N_S, it follows that $(\beta'', Tu)_\varphi = 0$ $\forall u \in D_T$. Consequently $\beta'' \in D_{T^*}$, and, moreover, $T^*\beta'' = 0$. Since the forms β'' and β are both in D_{T^*}, it follows that $\beta' \in D_{T^*}$, and therefore $\beta' \in D_{T^*} \cap N_S$. Substituting $\beta = \beta' + \beta''$ into (3.6.28) and recalling that $\alpha \in N_S$, we now see that (3.6.28) is valid for any form $\beta \in D_{T^*}$. Thus the linear functional $F(f)$ defined on the range of the operator T^* by

$$F(T^*\beta) = (\beta, \alpha)_\varphi^{(0,1)}$$

is continuous, and therefore can be extended without increasing its norm to the entire space $L^2(C^n, \varphi)$. By Riesz's theorem on the general form of a functional in Hilbert space, we then see that there exists an element f of $L^2(C^n, \varphi)$ such that

$$(f, T^*\beta)_\varphi = (\alpha, \beta)_\varphi \qquad \forall \beta \in D_{T^*}.$$

Consequently $f \in D_{T^{**}}$, $\alpha = T^{**}f$, and

$$\|f\|_\varphi = \|F\| \leqq \int_{C^n} |\alpha|^2 \frac{e^{-\varphi}}{c}\, d\omega. \tag{3.6.29}$$

[33] The notation in this subsection is carried over from the preceding subsection.

But the operator T is closed, and thus $T^{**} = T$. Therefore $\alpha = Tf$. Inequality (3.6.29) implies that

$$\int_{C^n} |f|^2 e^{-\varphi}\, d\omega \leqq \int_{C^n} |\alpha|^2 \frac{e^{-\varphi}}{c}\, d\omega.$$

Q.E.D.

THEOREM 3.6.2. *Let $\varphi(z)$ be an arbitrary plurisubharmonic function in C^n. Let the form $\alpha \in L^2_{(0,1)}(C^n, \varphi)$ be such that $\bar\partial\alpha = 0$. Then the equation $\bar\partial f = \alpha$ has a solution $f(z)$ in $L^2(C^n, \text{loc})$ such that*

$$\int_{C^n} |f|^2 e^{-\varphi}(1 + |z|^2)^{-2}\, d\omega_z \leqq \frac{1}{2} \int_{C^n} |\alpha|^2 e^{-\varphi}\, d\omega_z. \tag{3.6.30}$$

PROOF. Express the function $\varphi(z)$ as $\lim_{j\to\infty} \varphi_j(z)$, where the functions $\varphi_j(z)$ form a monotone decreasing sequence of infinitely differentiable plurisubharmonic functions. Since $\varphi_j \geqq \varphi$, it follows that $\alpha \in L^2(C^n, \varphi_j)$ for any j, a fortiori $\alpha \in L^2(C^n, \varphi_j + 2\ln(1 + |z|^2))$. Apply Theorem 3.6.1 to the function $\varphi_j(z) + 2\ln(1 + |z|^2)$, with $c(z) = 2(1 + |z|^2)^{-2}$. That this is possible follows from the inequality

$$\sum_{i,j=1}^{n} \zeta_i \bar\zeta_j \frac{\partial_2}{\partial z_i\, \partial \bar z_j} \ln(1 + |z|^2)$$
$$= (1 + |z|^2)^{-2}(|\zeta|^2(1 + |z|^2) - |\langle \zeta, z\rangle|^2) \geqq (1 + |z|^2)^{-2}|\zeta|^2. \tag{3.6.31}$$

We then see that for any $j = 1,2, \cdots$ there exists $f_j(z) \in L^2(C^n, \varphi_j + 2\ln(1 + |z|^2))$ such that $\bar\partial f_j = \alpha$ and

$$\int_{C^n} |f_j|^2 e^{-\varphi_j}(1 + |z|^2)^{-2}\, d\omega \leqq \frac{1}{2} \int_{C^n} |\alpha|^2 e^{-\varphi_j}\, d\omega \leqq \frac{1}{2} \int_{C^n} |\alpha|^2 e^{-\varphi}\, d\omega. \tag{3.6.32}$$

This inequality implies that the L^2 norms of the functions $f_j(z)$ are uniformly bounded in any bounded domain. Consequently in each such domain the sequence $\{f_j(z)\}_1^\infty$ is weakly compact, and one can therefore extract a subsequence $\{f_{j_n}(z)\}$ which converges weakly in every bounded domain to some function $f \in L^2(C^n, \text{loc})$. Weak convergence of these functions implies convergence of their derivatives, interpreted in the distribution-theoretic sense. Therefore it follows from the equalities $\bar\partial f_j = \alpha, j = 1,2, \cdots$, that $\bar\partial f = \alpha$. On the other hand, it follows from (3.6.32) that for any $m = 1,2, \cdots$ and any $t > 0$

$$\int_{|z|<t} |f|^2 e^{-\varphi_{j_n}}(1 + |z|^2)^{-2}\, d\omega_z \leqq \int_{C^n} |\alpha|^2 e^{-\varphi}\, d\omega_z. \tag{3.6.32'}$$

Indeed by the Schwarz-Bunjakovskiĭ inequality and inequality (3.6.32) we see that for $l, k > m$

$$\begin{aligned}
&\Big| \int_{|z|<t} f_{j_l} \bar{f}_{j_k} e^{-\varphi_{j_m}} (1 + |z|^2)^{-2}\, d\omega_z \Big| \\
&\quad \leqq \Big\{ \int_{|z|<t} |f_{j_l}|^2 e^{-\varphi_{j_m}} (1 + |z|^2)^{-2}\, d\omega_z \Big\}^{1/2} \Big\{ \int_{|z|<t} |f_{j_k}|^2 e^{-\varphi_{j_m}} (1 + |z|^2)^{-2}\, d\omega_z \Big\}^{1/2} \\
&\quad \leqq \Big\{ \int_{|z|<t} |f_{j_l}|^2 e^{-\varphi_{j_l}} (1 + |z|^2)^{-2}\, d\omega_z \Big\}^{1/2} \Big\{ \int_{|z|<t} |f_{j_k}|^2 e^{-\varphi_{j_k}} (1 + |z|^2)^{-2}\, d\omega_z \Big\}^{1/2} \\
&\quad \leqq \int_{C^n} |\alpha|^2 e^{-\varphi}\, d\omega.
\end{aligned}$$

Hence, since the functions f_{j_l} and f_{j_k} converge weakly to f, we get inequality (3.6.32′). Letting $m \to \infty$ in (3.6.32′), we get the estimate (3.6.30), proving the theorem.

THEOREM 3.6.3. *Let $\varphi(z)$ be a plurisubharmonic function in C^n such that for any z' and z with $|z' - z| < 1$*

$$|\varphi(z') - \varphi(z)| < c \tag{3.6.33}$$

for some constant c. Let Σ be an analytic plane in C^n, of complex dimension l,[34] and $f(z)$ a function analytic on Σ such that

$$\int_{\Sigma} |f|^2 e^{-\varphi}\, d\sigma < \infty \tag{3.6.34}$$

(where $d\sigma$ is Lebesgue measure on Σ). Then there exists a function F analytic in C^n such that $F(z) = f(z)$ on Σ and

$$\int_{C^n} |F|^2 e^{-\varphi} (1 + |z|^2)^{-3(n-l)}\, d\omega_z \leqq A \int_{\Sigma} |f|^2 e^{-\varphi}\, d\sigma, \tag{3.6.35}$$

where the constant A depends on n, l and c but not on the function f.[35]

PROOF. It has already been proved that the function $\ln(1 + |z|^2)$ is plurisubharmonic in C^n. It is also readily seen that this function satisfies condition (3.6.33) with $c = \ln 3$. Our theorem need be proved, therefore, only for $n = l + 1$, and the case of arbitrary n will then follow by induction. Thus let $n = l + 1$. Without loss of generality we may assume $\Sigma = \{z\colon z \in C^{l+1},\ z_{l+1} = 0\}$. Then f is an analytic function of $z' = (z_1, \cdots, z_l)$, and by (3.6.33)

$$\int_{|z_{l+1}|<1} |f|^2 e^{-\varphi(z)}\, d\omega_z \leqq \pi e^c \int_{\Sigma} |f|^2 e^{-\varphi(z)}\, d\sigma_{z'}. \tag{3.6.36}$$

Now take any function $\psi(w)$, $w \in C^1$, continuously differentiable in C^1, equal to 1 for $|w| < 1/2$, vanishing for $|w| > 1$, and such that $|\partial \psi / \partial \bar{w}| \leqq 3/2$ throughout C^1 and $|\psi| \leqq 1$. Set

$$F(z) = \psi(z_{l+1}) f(z') - z_{l+1} u(z), \tag{3.6.37}$$

[34] In other words,

$$\Sigma = \{z\colon z_i = a_{i,1} w_1 + \cdots + a_{i,l} w_l + a_{i,l+1},\ (w_1, \cdots, w_l) \in C^l\},$$

where $a_{i,j}$ are arbitrary fixed complex numbers and the rank of the matrix $\{a_{i,j}\}_{i \leqq n,\, j \leqq l}$ is l.

[35] For theorems on continuation with estimates, see also Palamodov [1].

where $u(z) \in L^2(C^n, \text{loc})$. Note that $F(z) = f(z')$ for $z_{l+1} = 0$. We claim that the function $u(z)$ can be so chosen that $F(z)$ is the required function. Since $F(z)$ must be analytic, we must have $\bar\partial F = 0$ throughout C^n. In view of (3.6.37) it follows from this condition that

$$\bar\partial u = z_{l+1}^{-1} f(z')\bar\partial\psi(z_{l+1}) = z_{l+1}^{-1} f(z') \frac{\partial\psi}{\partial\bar z_{l+1}}\, d\bar z_{l+1}.$$

A direct check shows that

$$\bar\partial\Big(z_{l+1}^{-1} f(z') \frac{\partial\psi}{\partial\bar z_{l+1}}\, d\bar z_{l+1}\Big) = 0.$$

We further observe the following inequality, which follows from (3.6.36) and the properties of $\psi(w)$:

$$\int_{C^{l+1}} \Big|z_{l+1}^{-1} f(z') \frac{\partial\psi}{\partial\bar z_{l+1}}\, d\bar z_{l+1}\Big|^2 e^{-\varphi}\, d\omega_z = \int_{|z_{l+1}|<1} \Big|z_{l+1}^{-1} f(z') \frac{\partial\psi}{\partial\bar z_{l+1}}\Big|^2 e^{-\varphi}\, d\omega_z$$
$$\leqq 9 \int_{|z_{l+1}|<1} |f(z')|^2 e^{-\varphi}\, d\omega_z \leqq 9\pi e^c \int_\Sigma |f(z')|^2 e^{-\varphi}\, d\sigma.$$

Thus with $\alpha = z_{l+1}^{-1} f(z')\,\bar\partial\psi$ we can apply Theorem 3.6.2. By this theorem there exists a function $u \in L^2(C^{l+1}, \text{loc})$ satisfying the condition $\bar\partial u = \alpha$, such that

$$\int_{C^{l+1}} |u|^2 e^{-\varphi}(1 + |z|^2)^{-2}\, d\omega_z \leqq \frac{9}{2}\pi e^c \int_\Sigma |f|^2 e^{-\varphi}\, d\sigma. \tag{3.6.38}$$

The function $F(z)$ defined in terms of u by (3.6.37) satisfies the condition $\bar\partial F = 0$. Therefore the class of equivalent functions (i.e. equal to F almost everywhere) contains an entire function, and therefore we may assume without loss of generality that $F(z)$ is an entire function.[36] Then by (3.6.38) and (3.6.36) we easily see that $F(z)$ satisfies the estimate

$$\int_{C^{l+1}} |F|^2(1 + |z|^2)^{-3} e^{-\varphi}\, d\omega_z \leqq 11\pi e^c \int_\Sigma |f|^2 e^{-\varphi}\, d\sigma.$$

Q.E.D.

THEOREM 3.6.4. *Let $\varphi(z)$ be a twice continuously differentiable plurisubharmonic function in C^n. Then the set of entire functions $f(z)$, $z \in C^n$, such that for some integer $N = N(f) > 0$*

$$\int_{C^n} |f|^2 e^{-\varphi}(1 + |z|^2)^{-N}\, d\omega_z < \infty \tag{3.6.39}$$

is dense in the space $A(C^n)$ of entire functions, in the topology of uniform convergence on compact subsets.

[36] In Chapter 1, § 2, it was shown that a distribution f such that $\Delta f = 0$ can be identified in a natural way with a classical harmonic function. The fact that any distribution F, in particular, $F \in L^2(C^n, \text{loc})$, such that $\bar\partial F = 0$ may be regarded without loss of generality as an entire function is an assertion of the same type, and it follows from the above as a corollary.

PROOF. The theorem will be proved if we can show that any continuous linear functional $\mu(f)$ on $A(C^n)$ which vanishes on entire functions satisfying condition (3.6.39) also vanishes on any function of $A(C^n)$. Note that, as follows from the properties of subharmonic functions, the topology of uniform convergence on all compact subsets is equivalent to the topology of L^2 convergence on all compact subsets. The space $A(C^n)$ may therefore be treated as a countably normed space[37] with system of norms

$$\|f\|_m = \Big(\int_{|z|<m} |f|^2 \, d\omega_z\Big)^{1/2}.$$

It is known that a functional $\mu(f)$ continuous in the topology of a countably normed space is also continuous in one of the norms $\|f\|_m$. Consequently by the Hahn-Banach Theorem there exists $m_0 = m_0(\mu)$ such that $\mu(f)$ can be extended as a continuous linear functional to the space $L^2(E_{m_0})$, where $E_{m_0} = \{z: |z| < m_0\}$ and $L^2(E_{m_0}) = L^2(E_{m_0}, 0)$. Thus there exists a function $v(z) \in L^2(E_{m_0})$ such that

$$\mu(f) = \int_{E_{m_0}} f\bar{v} \, d\omega \qquad \forall f \in L^2(E_{m_0}). \tag{3.6.39'}$$

Complete the function $v(z)$ by defining it as zero outside the ball E_{m_0}. It then follows from (3.6.39') that

$$\mu(f) = \int_{C^n} f\bar{v} \, d\omega \qquad \forall f \in A(C^n).$$

Let $\chi(t)$ be any twice continuously differentiable function, convex and monotone nondecreasing on the entire real axis, vanishing on the set $(-\infty, \ln(1 + R^2))$, where $R > m_0$, and such that $\chi(t) = t + \text{const}$ for sufficiently large t. Set

$$\varphi_N(z) = \varphi(z) + \ln(1 + |z|^2) + N\chi(\ln(1 + |z|^2)).$$

It is readily seen that, by (3.6.31),

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi_N}{\partial z_i \partial \bar{z}_j} \zeta_i \bar{\zeta}_j \geqq (1 + |z|^2)^{-2} |\zeta|^2. \tag{3.6.40}$$

Now fix N and consider the spaces $L^2(C^n, \varphi_N)$, $L^2_{(0,1)}(C^n, \varphi_N)$, $L^2_{(0,2)}(C^n, \varphi_N)$ and the corresponding operators T and S. The elements of the null space N_T of the operator T, i.e. the functions $f(z) \in L^2(C^n, \varphi_N)$ such that $\bar{\partial} f = 0$, are entire functions satisfying condition (3.6.39). Hence the assumption that the functional $\mu(f)$ vanishes on entire functions satisfying (3.6.39) implies that the function $v(z)e^{\varphi_N(z)}$ is orthogonal to the space N_T. Hence it follows that the function $v(z)e^{\varphi_N(z)}$ lies in the closure of the range of the operator T^*. Consider a sequence of forms $\alpha^{(j,N)}$ in $L^2_{(0,1)}(C^n, \varphi_N)$ such that the functions $u_{j,N} = T^*\alpha^{(j,N)}$ converge in $L^2(C^n, \varphi_N)$ to ve^{φ_N}.

Without loss of generality we may assume that the forms $\alpha^{(j,N)}$ are orthogonal to the null space of the operator T^*, and hence belong to the closure of the range of T,

[37] For details on countably normed spaces, see, for example, Gel'fand and Šilov [2].

therefore also to the null space of the operator S. Therefore, applying Lemma 3.6.3 with $c(z) = (1 + |z|^2)^{-2}$, which is possible by (3.6.40), we get the estimate

$$\int_{C^n} |\alpha^{(j,N)}|^2 (1 + |z|^2)^{-2} e^{-\varphi_N} d\omega_z \leqq \int_{C^n} |u_{j,N}|^2 e^{-\varphi_N} d\omega_z. \tag{3.6.41}$$

This estimate is clearly valid not only for $u_{j,N}$ and $\alpha^{(j,N)}$, but also for any form $\alpha \in D_{T^*} \cap N_S$ and function $u = T^*\alpha$. Hence the convergence of the functions $u_{j,N}$ in $L^2(C^n, \varphi_N)$ to the function ve^{φ_N} implies that the forms $\alpha^{(j,N)}$ converge in the space $L^2_{(0,1)}(C^n, \text{loc})$ to some form $\alpha^{(N)} \in L^2_{(0,1)}(C^n, \varphi_{N+2})$. On the other hand, convergence of the forms $\alpha^{(j,N)}$ in $L^2_{(0,1)}(C^n, \text{loc})$ to the form $\alpha^{(N)}$ implies that the coefficients $e^{-\varphi_N} \alpha_i^{(j,N)}$ of the forms $e^{-\varphi_N} \alpha^{(j,N)}$ converge to the coefficients $e^{-\varphi_N} \alpha_i^{(N)}$ of the form $e^{-\varphi_N} \alpha^{(N)}$ in the distribution-theoretic sense. But then the functions $(\partial/\partial z_i)\{\alpha_i^{(j,N)} e^{-\varphi_N}\}$ converge as $j \to \infty$ to $(\partial/\partial z_i)\{\alpha_i^{(N)} e^{-\varphi_N}\}$, again as distributions. Now, by (3.6.9), the equation $T^*\alpha^{(j,N)} = u_{j,N}$ may be written as

$$\sum_{i=1}^{n} \frac{\partial}{\partial z_i} \{\alpha_i^{(j,N)} e^{-\varphi_N}\} = - u_{j,N} e^{-\varphi_N}. \tag{3.6.42}$$

The convergence of the functions $e^{-\varphi_N} \alpha_i^{(j,N)}$ and their derivatives implies that

$$\sum_{i=1}^{n} \frac{\partial}{\partial z_i} (\alpha_i^{(N)} e^{-\varphi_N}) = - v,$$

and moreover, by (3.6.41),

$$\int_{C^n} |\alpha^{(N)}|^2 (1 + |z|^2)^{-2} e^{-\varphi_N} d\omega_z \leqq \int_{C^n} |v|^2 e^{\varphi_N} d\omega_z.$$

Since $\varphi_N(z) = \varphi(z) + \ln(1 + |z|^2)$ for $|z| < R$, and $v(z) = 0$ for $|z| > R$, the right-hand side of this inequality is independent of N. Thus the sequence of forms $\beta^{(N)} = \alpha^{(N)} e^{-\varphi_N}$ satisfies the conditions

$$\sum_{i=1}^{n} \frac{\partial \beta_i^{(N)}}{\partial z_i} = - v \tag{3.6.43}$$

and

$$\int_{C^n} |\beta^{(N)}|^2 (1 + |z|^2)^{-2} e^{\varphi_N} d\omega_z \leqq C, \tag{3.6.44}$$

where C is a constant independent of N.

Since a bounded subset of a Hilbert space is weakly compact, it follows from (3.6.44) that there exists a subsequence of forms $\beta^{(N_i)}$ which is weakly convergent in $L^2_{(0,1)}(C^n, \psi)$, where $\psi = - \varphi + 2 \ln (1 + |z|^2)$, to some form $\beta \in L^2_{(0,1)}(C^n, \psi)$. Moreover, for every N this form satisfies the inequality

$$\int_{C^n} |\beta|^2 (1 + |z|^2)^{-2} e^{\varphi_N} d\omega_z \leqq C, \tag{3.6.44'}$$

which can be proved in the same way as (3.6.32').

Since the functions φ_N, which are monotone increasing, tend to $+\infty$ when $|z| > R$, it follows from (3.6.44') that $\beta = 0$ for $|z| > R$. Weak convergence of the

forms $\beta^{(N_i)}$ in $L^2(C^n, \psi)$ obviously implies the convergence of their coefficients, and also of their derivatives, in the distribution-theoretic sense. In the same sense, therefore, it follows from (3.6.43) that in the limit

$$\sum_{i=1}^{n} \frac{\partial \beta_i}{\partial z_i} = -\upsilon.$$

In other words, for any function $u \in D(C^n)$,

$$\int_{C^n} u\bar{\upsilon}\, d\omega_z = -\int_{C^n} u \sum_{i=1}^{n} \frac{\partial \bar{\beta}_i}{\partial \bar{z}_i}\, d\omega_z = \sum_{i=1}^{n} \int_{C^n} \frac{\partial u}{\partial \bar{z}_i}\, \bar{\beta}_i\, d\omega_z. \tag{3.6.45}$$

Since the functions β_i and υ have compact supports, equality (3.6.45) is also valid for all infinitely differentiable functions $u(z)$, among these the functions $u(z) \in A(C^n)$. But $\partial u/\partial \bar{z}_i \equiv 0$ $\forall u \in A(C^n)$, $i = 1, \cdots, n$. Consequently

$$\mu(u) = \int_{C^n} u\bar{\upsilon}\, d\omega_z = 0$$

for any function $u \in A(C^n)$. Q.E.D.

3. *Analytic functionals.* As before, let $A(C^n)$ be the space of entire functions in C^n with the topology of uniform convergence on all compact subsets. Let A' denote the dual space of $A(C^n)$. The elements of A', i.e. the continuous linear functions on $A(C^n)$, will be called *analytic functionals.*[38] A compact set K is said to *define an analytic functional* μ if for any neighborhood Ω of K there exists a constant C_Ω such that

$$|\mu(f)| \leqq C_\Omega \sup_{\Omega} |f| \qquad \forall f \in A(C^n).$$

It follows from the representation (3.6.39′) that every analytic functional has at least one defining set. In general, however, there does not exist a minimal defining set, i.e. a defining set which is a subset of all other defining sets of the analytic functional in question.

Consider the closure in $A(C^n)$ of the linear span of the set of exponentials $\exp\{\langle z, \zeta\rangle\}$, where $\zeta \in C^n$. It is clear that this closure contains all derivatives of the functions $\exp\{\langle z, \zeta\rangle\}$ with respect to ζ. Consequently it also contains all monomials z^k which are derivatives of such exponentials at $\zeta = 0$. Hence it follows that the closure in question is identical with the space $A(C^n)$, and so the linear combinations of these exponentials constitute a dense subset of $A(C^n)$. Therefore any analytic function is uniquely determined by its values on exponential functions. We now define the *Laplace transform of an analytic functional* μ to be the function

$$\tilde{\mu}(\zeta) = \mu_z(\exp\langle z, \zeta\rangle). \tag{3.6.46}$$

Since the functional $\mu(f)$ is linear and continuous, we can differentiate with respect

[38] Results bearing on the theory of analytic functionals and not presented here may be found, e.g., in Kiselman [1], [2], Martineau [1], and Ehrenpreis [1].

to ζ "inside" the functional. Thus the function $\tilde{\mu}(\zeta)$ is an entire function in C^n. In addition, it is an entire function of exponential type. Indeed, as already mentioned, every functional μ has a defining set K, and hence for every neighborhood Ω of K there is a constant C_Ω such that

$$|\tilde{\mu}(\zeta)| \leqq C_\Omega \exp\left\{\sup_{z\in\Omega} \operatorname{Re}\langle z, \zeta\rangle\right\} \qquad \forall \zeta \in C^n. \tag{3.6.47}$$

It follows from (3.6.47) that for some sufficiently large $\sigma > 0$

$$|\tilde{\mu}(\zeta)| \leqq C_\Omega e^{\sigma|\zeta|} \qquad \forall \zeta \in C^n \tag{3.6.48}$$

and thus $\tilde{\mu}(\zeta)$ is a function of exponential type. The number σ figuring in (3.6.48) is obviously related to the "size" of the defining set K. A more rigorous description of the relationship between the growth of Laplace transforms of analytic functions and their defining sets is given by the following theorem (see Martineau [1] and Ehrenpreis [1]).

THEOREM 3.6.5. *If $\mu \in A'$ and K is a defining set for μ, then for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$|\tilde{\mu}(\zeta)| \leqq C_\varepsilon \exp\{H_K(\zeta) + \varepsilon|\zeta|\} \qquad \forall \zeta \in C^n, \tag{3.6.49}$$

where

$$H_K(\zeta) = \sup_{z\in K} \operatorname{Re}\langle z, \zeta\rangle,$$

i.e. $H_K(\zeta)$ is the support function of the convex hull of the set $K' = \{\zeta : \bar{\zeta} \in K\}$. Conversely, if $\Phi(\zeta)$ is an entire function and K a convex compact set such that for any $\varepsilon > 0$

$$|\Phi(\zeta)| \leqq C_\varepsilon \exp\{H_K(\zeta) + \varepsilon|\zeta|\} \qquad \forall \zeta \in C^n, \tag{3.6.50}$$

then there exists an analytic functional μ with defining set K such that $\tilde{\mu}(\zeta) = \Phi(\zeta)$.

PROOF. The estimate (3.6.49) follows immediately from (3.6.47). To prove the existence of an analytic functional whose Laplace transform is a given function $\Phi(\zeta)$ satisfying condition (3.6.50), we first consider the case $n = 1$. In this case it follows from the estimate (3.6.49) by Pólya's theorem (Theorem 3.3.2) that the function $F(\zeta)$, $\zeta \in C^1$, associated with $\Phi(\zeta)$ is holomorphic outside the compact set K. Hence the contour l in the integral representation (see § 3)

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_l e^{z\zeta} F(z)\, dz$$

may be any contour enclosing the set K. It is now clear that the required functional is the functional $\mu(f)$ defined by

$$\mu(f) = \frac{1}{2\pi i} \int_l f(z)\, F(z)\, dz.$$

Now consider the case of arbitrary n. Set $K_\varepsilon = \{\zeta\colon \operatorname{Re}\langle z, \zeta\rangle \leqq H_K(z) + \varepsilon|z| \ \forall z \in C^n\}$. To prove the assertion it will suffice to show that for any $\varepsilon > 0$ a function $\Phi(\zeta)$ satisfying (3.6.50) admits the representation

$$\Phi(\zeta) = \int_{C^n} \psi_\varepsilon(z)\, e^{\langle z,\zeta\rangle}\, d\omega_z, \tag{3.6.51}$$

where $\psi_\varepsilon(z)$ is a function continuous in C^n which vanishes off K_ε.

Denoting $z_j = t_{2j-1} + it_{2j}$, set

$$\varphi_\varepsilon(\lambda) = \sup_{z\in K_\varepsilon}\left\{\sum_{j=1}^{2n} t_j \operatorname{Im}\lambda_j\right\},$$

where $\lambda \in C^{2n}$. It is readily seen that $\varphi_\varepsilon(\lambda)$ is the support function of the convex compact set $K_\varepsilon'' \subset C^{2n}_{(\lambda)}$ which is the image of the compact set $K_\varepsilon \subset C^n_{(z)}$ under the mapping $C^{(n)}_{(z)} \to C^{2n}_{(\lambda)} \cap \{\lambda\colon \operatorname{Re}\lambda_1 = 0, \cdots, \operatorname{Re}\lambda_{2n} = 0\}$ defined by

$$\operatorname{Re} z_j = \operatorname{Im}\lambda_{2j-1}, \qquad \operatorname{Im} z_j = \operatorname{Im}\lambda_{2j}, \qquad j = 1, \cdots, n.$$

On $\Sigma = \{\lambda\colon \lambda_1 = i\zeta_1, \lambda_2 = -\zeta_1, \cdots, \lambda_{2n} = -\zeta_n\}$ the function $\varphi_\varepsilon(\lambda)$ transforms as follows:

$$\begin{aligned}\varphi_\varepsilon(\lambda)|_\Sigma &= \sup_{z\in K_\varepsilon}\{t_1 \operatorname{Re}\zeta_1 - t_2 \operatorname{Im}\zeta_1 + \cdots - t_{2n}\operatorname{Im}\zeta_n\}\\ &= \sup_{z\in K_\varepsilon} \operatorname{Re}\langle z, \zeta\rangle = H_K(\zeta) + \varepsilon|\zeta|.\end{aligned}$$

Hence the function $u(\lambda)$ defined on Σ by the condition $u(\lambda)|_{\lambda\in\Sigma} = \Phi(\zeta)$ satisfies the inequality

$$\int_\Sigma |u(\lambda)|^2\, e^{-2\varphi_\varepsilon}\, d\sigma < \infty.$$

Since $\varphi_\varepsilon(\lambda)$ is the support function of the compact set K_ε'', it is a convex function, and therefore also plurisubharmonic. Moreover, as follows from the properties of support functions, $\varphi_\varepsilon(\lambda)$ is uniformly continuous in C^{2n}. Consequently (see Theorem 3.6.3) there exists an entire function $U(\lambda)$ in C^{2n} such that $U(\lambda) = u(\lambda)$ on Σ and

$$\int_{C^{2n}} |U(\lambda)|^2\, e^{-2\varphi_\varepsilon(\lambda)}(1 + |\lambda|^2)^{-3n}\, d\omega_\lambda < \infty. \tag{3.6.52}$$

Since the function $|U(\lambda)|$ is subharmonic, it follows from (3.6.52) that for any $\lambda \in C^{2n}$

$$|U(\lambda)| \leqq \text{const} \cdot e^{\varphi_\varepsilon(\lambda)}(1 + |\lambda|^2)^{3n/2}. \tag{3.6.53}$$

Now set $\psi = U(\lambda)\tilde\chi(\lambda)$, where $\tilde\chi(\lambda)$ is the Fourier transform of some infinitely differentiable function $\chi(z) = \chi(t)$[39] such that $\chi(z) = 0$ for all $z \in \{z\colon |z| > \varepsilon\}$, $\chi(z) = \chi(|z|)$ and $\int_{C^n}\chi(z)\, d\omega_z = 1$. It is obvious that $\tilde\chi(0) = 1$ and that $\tilde\chi(\lambda)$ is a function of $\sum_1^{2n}\lambda_i^2$. Therefore $\chi(\lambda) = 1$ on Σ, and consequently the function

[39] Recall that $\operatorname{Re} z_j = t_{2j-1}$ and $\operatorname{Im} z_j = t_{2j}$.

$\psi(\lambda)$ is identical on Σ with $u(\lambda)$, or, equivalently, with the given function $\Phi(\zeta)$. Note moreover that for any integer N

$$|\tilde{\chi}(\lambda)| \leqq \text{const} \cdot (1 + |\lambda|)^{-N} \exp\{\varepsilon \sqrt{\textstyle\sum_1^n (\operatorname{Im} \lambda_i)^2}\}. \tag{3.6.54}$$

Indeed, by definition

$$\tilde{\chi}(\lambda) = \int_{C^n} \chi(z) \exp\left\{i \sum_{j=1}^{n} (\lambda_{2j-1} \operatorname{Re} z_j + \lambda_{2j} \operatorname{Im} z_j)\right\} d\omega_z = \int_{R^{2n}} e^{\langle \lambda, t\rangle i} \chi(t)\, d\omega_t.$$

Integrating the last term by parts N times with respect to each variable, we see that for $\lambda_j \neq 0$

$$|\tilde{\chi}(\lambda)| \leqq \text{const} \cdot \frac{\exp\{\varepsilon \sqrt{\sum_1^{2n} |\operatorname{Im} \lambda_i|^2}\}}{|\lambda_j|^N} \qquad (j = 1, \cdots, 2n),$$

whence (3.6.54) follows immediately.

It now follows from (3.6.54) and (3.6.53) that for real values of $\lambda_1, \cdots, \lambda_{2n}$ the function $\psi(\lambda)$ is in $L^2(R^{2n})$ and its P-indicator does not exceed $\varphi_{2\varepsilon}(\lambda)$. Thus we may apply Theorem 3.4.2, and the result is the representation

$$\psi(\lambda) = \int_{C^n} \psi_\varepsilon(z) \exp\left\{- i \sum_{j=1}^{n} \lambda_{2j-1} \operatorname{Re} z_j - i \sum_{j=1}^{n} \lambda_{2j} \operatorname{Im} z_j\right\} d\omega_z, \tag{3.6.55}$$

where $\psi_\varepsilon(z) \in L^2(C^n)$ vanishes off $K_{2\varepsilon}$. Moreover, as follows from (3.6.54) and (3.6.53), the function $\psi(\lambda)$ decreases for $|\lambda| \to \infty$, $\lambda \in R^{2n}$, faster than $(1/|\lambda|)^N$, $N > 0$, and is otherwise arbitrary; hence one readily sees that $\psi_\varepsilon(z)$ is not only in $L^2(C^n)$ but also infinitely differentiable. Setting $\lambda_{2j-1} = i\zeta_j$, $\lambda_{2j} = -\zeta_j$ in (3.6.55), and recalling that $\psi(\lambda) = \Phi(\zeta)$ on Σ, we get the representation (3.6.51) for the function $\Phi(\zeta)$, thus completing the proof of the theorem.

4. *Radial indicator.* We shall now prove Theorem 3.5.1, which states that there exists an entire function with given indicator $L_r^*(z)$. In proving this theorem we shall use the concept of the radial indicator of a plurisubharmonic function, using the same definition and notation as for entire functions:

$$L_r(z; \psi) = \overline{\lim_{t \to +\infty}} \frac{1}{t^\rho} \psi(tz).$$

Here $\psi(z)$ is a plurisubharmonic function in C^n of finite order $\rho > 0$. Clearly, as for entire functions, the regularization $L_r^*(z; \psi)$ of the indicator $L_r(z; \psi)$ of a plurisubharmonic function of finite order ρ is plurisubharmonic in C^n and positively homogeneous of degree ρ.

We shall need certain transformations of plurisubharmonic functions which leave the indicator $L_r^*(z; \psi)$ invariant.

LEMMA 3.6.4. *Let $\psi(z)$ be a plurisubharmonic function (in C^n) of finite order $\rho > 0$. Let*

$$\Psi(z) = \sup_{|\zeta_1|,\cdots,|\zeta_n|<a} \phi(z_1 + \zeta_1, \cdots, z_i + \zeta_i, z_{i+1}, \cdots, z_n),$$

and let $\Psi^*(z)$ *be the regularization of* $\Psi(z)$. *Then for any* $a > 0$ *and* $i = 1, \cdots, n$,

$$L_r^*(z; \Psi^*) = L_r^*(z; \phi).$$

PROOF. Let $g(z)$ be any continuous positively homogeneous function of degree ρ, majorizing the function $L_r^*(z; \phi)$. As in (g) of § 5, one proves that for any $\varepsilon > 0$ and some constant $c(\varepsilon)$ the inequality $\phi(z) \leqq c(\varepsilon) + g(z) + \varepsilon|z|^\rho$ is valid. It follows from this inequality that

$$\Psi^*(z) \leqq c(\varepsilon) + \varepsilon(|z| + na)^\rho + \sup_{|\zeta_1|,\cdots,|\zeta_n|<a} g(z_1 + \zeta_1, \cdots, z_i + \zeta_i, z_{i+1}, \cdots, z_n). \tag{3.6.56}$$

Since $g(z)$ is continuous and positively homogeneous,

$$\sup_{|\zeta_1|,\cdots,|\zeta_n|<a} g(z_1 + \zeta_1, \cdots, z_i + \zeta_i, z_{i+1}, \cdots, z_n) \leqq g(z) + \delta(z)|z|^\rho,$$

where $\delta(z)$ is some function which tends to zero as $z \to \infty$. It therefore follows from (3.6.56) that $L_r(z; \Psi^*) \leqq g(z) + \varepsilon|z|^\rho$ for any $\varepsilon > 0$. Consequently $L_r^*(z; \Psi^*) \leqq g(z)$, and since the function $L_r^*(z; \phi)$ is clearly the limit of functions $g(z)$ of the above type, $L_r^*(z; \Psi^*) \leqq L_r^*(z; \phi)$. The converse inequality is obvious. Thus $L_r^*(z; \Psi^*) = L_r^*(z; \phi)$. Q.E.D.

LEMMA 3.6.5. *Let* $\phi(z)$ *be a plurisubharmonic function (in* C^n*) of finite order* $\rho > 0$, *and let* $\phi_\eta(z) = \phi * \eta$, *where* $\eta(z) \in D(C^n)$ *is a function of* $|z|$ *satisfying the conditions* $0 \leqq \eta(z) \leqq 1$ *and* $\int_{C^n} \eta(z)dz = 1$. *Then* $L_r^*(z; \phi_\eta) = L_r^*(z; \phi)$.

PROOF. It is clear that for $i = n$ and some $a > 0$ the function $\Psi(z)$ defined in Lemma 3.6.4 is related to the function $\phi_\eta(z)$ by $\phi_\eta(z) \leqq \Psi(z)$. Consequently

$$L_r(z; \phi_\eta) \leqq L_r(z; \Psi) \leqq L_r^*(z; \phi). \tag{3.6.57}$$

On the other hand (see Chapter 1, § 1. 6), we have $\phi(z) \leqq \phi_\eta(z)$ everywhere in C^n, and so

$$L_r(z; \phi) \leqq L_r(z; \phi_\eta). \tag{3.6.58}$$

It follows from (3.6.57) and (3.6.58) that $L_r^*(z; \phi) = L_r^*(z; \phi_\eta)$. Q.E.D.

We now prove a weakened version of Theorem 3.5.1.

THEOREM 3.6.6 (MARTINEAU [2], [3]). *Let* $\varphi(z)$ *be a positively homogeneous plurisubharmonic function in* C^n *of degree* $\rho > 0$. *Then for every point* $z^0 \neq 0$ *there exists an entire function* $f_{z^0}(z)$ *of finite order* ρ *such that* $L_r^*(z; f_{z^0}) \leqq \varphi(z)$ *for all* $z \in C^n$, and $L_r^*(z^0; f_{z^0}) = \varphi(z^0)$.

The proof of this theorem is based on the following lemmas.

LEMMA 3.6.6. *If $f(z)$, $z \in C^n$, is an entire function such that*

$$\int_{C^n} |f(z)|^2 e^{-2\psi(z)}\, d\omega_z < \infty,$$

where $\psi(z)$ is a plurisubharmonic function of finite order, then

$$L_r(z; f) \leqq L_r^*(z; \psi).$$

PROOF. Let $\tau(t)$ be a nonnegative infinitely differentiable function in R_+^1, vanishing for $t < 1$ and $t > 2$, and such that $\int_0^\infty t\tau(t)\, dt = 1/2\pi$. Set

$$\eta(z) = \tau(|z_1|)\, \tau(|z_2|) \cdots \tau(|z_n|).$$

By the mean value theorem for harmonic functions,

$$f(z) = \int_{C^n} f(z + \zeta)\eta(\zeta)\, d\omega_\zeta.$$

We rewrite this equality as

$$f(z) = \int_{C^n} f(z + \zeta)\, e^{-\psi(z+\zeta)}\, \eta(\zeta)\, e^{\psi(z+\zeta)}\, d\omega_\zeta$$

and estimate the integral using the Schwarz-Bunjakovskiĭ inequality:

$$|f(z)| \leqq M \|\eta(\zeta)\, e^{\psi(z+\zeta)}\|_{L^2}, \tag{3.6.59}$$

where $M = \|f e^{-\psi}\|_{L^2}$. Further,

$$\|\eta(\zeta)\, e^{\psi(z+\zeta)}\|_{L^2} \leqq (4\pi)^n M_1\, e^{\Psi(z)}, \tag{3.6.59'}$$

where

$$M_1 = \max_{C^n} |\eta(\zeta)| < \infty, \qquad \Psi(z) = \sup_{\substack{|\zeta_i| \leq 2 \\ i=1,\cdots,n}} \psi(z + \zeta).$$

It follows from (3.6.59) and (3.6.59′) that $L_r(z; f) \leqq L_r(z; \Psi^*)$. But by Lemma 3.6.4 we have $L_r^*(z; \Psi^*) = L_r^*(z; \psi)$. Consequently $L_r(z; f) \leqq L_r^*(z; \psi)$. Q.E.D.

LEMMA 3.6.7. *For any nonnegative function $\gamma(r)$, $0 \leqq r < \infty$, which tends to zero as $r \to \infty$ there exists a function $\Phi(r)$ such that for given $\rho > 0$*

$$\Phi(r) \geqq \gamma(r)\, r^\rho, \qquad \lim_{r\to\infty} \frac{\Phi(r)}{r^\rho} = 0, \tag{3.6.60}$$

and such that the function $\Phi(|z|)$, $z \in C^n$, is infinitely differentiable and strictly plurisubharmonic.

PROOF. It was shown in Chapter 1 that convex functions in R^m are also subharmonic. Hence it immediately follows that convex functions in C^n are plurisubharmonic. A natural approach to the proof of our lemma is thus to look for a suitable convex function. This is easily done if $\rho > 1$. Indeed, in this case we take

any function $\beta(t)$, $-\infty < t < \infty$, which is convex, infinitely differentiable, and such that $|\beta'(t)| < \rho - 1$, $\beta(t) \geqq \ln \gamma(e^t)$ and $\lim_{t\to+\infty} \beta(t) = -\infty$. The existence of such a function is obvious. Now set $\Phi_1(r) = (\exp \beta(\ln r))r^\rho$. The function $\Phi_1(r)$ obviously satisfies conditions (3.6.60). At the same time,

$$d\Phi_1/dr = e^{\beta(\ln r)}\, r^{\rho-1}(\beta' + \rho) > 0,$$
$$d^2\Phi_1/dr^2 = e^{\beta(\ln r)}\, r^{\rho-2}\,[\beta'' + (\beta' + \rho)(\beta' + \rho - 1)] \geqq 0,$$

and thus $\Phi_1(r)$ is convex and monotone increasing. Obviously the values of the function $\Phi_1(r)$ in the neighborhood of the origin may be so increased that the resulting function $\Phi_2(r)$ is still convex, infinitely differentiable, monotone nondecreasing, and $\Phi_2(r) \equiv \Phi_2(0)$ in a neighborhood of the origin. The function $\Phi_2(|z|)$ is then infinitely differentiable, convex, and therefore plurisubharmonic in C^n. It is now clear that the function we require may be any function

$$\Phi(r) = \Phi_2(r) + \lambda \ln(1 + r^2), \qquad \lambda > 0.$$

The term $\lambda \ln(1 + r^2)$ guarantees that $\Phi(r)$ will be strictly plurisubharmonic.

For $\rho \leqq 1$, we first construct a convex infinitely differentiable function $\alpha(t)$ such that $\alpha(t) \to 0$ as $t \to \infty$, $\alpha(t) \geqq \gamma(t^{1/2})$ for all $t \in (0, \infty)$, and for all $t > 0$

$$t\alpha' + \frac{\rho^2}{4(\rho + 1)}\,\alpha \geqq 0. \tag{3.6.61}$$

It is readily seen that this is possible. Now set $\Phi_1(r) = \alpha(r^2)\, r^\rho$. The function $\Phi_1(|z|)$ is plurisubharmonic.[40] Indeed, calculating the quadratic form

$$L(\Phi_1; z, \zeta) = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi_1(|z|)}{\partial z_i\, \partial \bar{z}_j}\, \zeta_i \bar{\zeta}_j,$$

we get

$$L(\Phi_1; z, \zeta) = \left\{\alpha''(|z|^2)|z|^\rho + \rho|z|^{\rho-1}\alpha'(|z|^2) + \frac{\rho}{2}\left(\frac{\rho}{2} - 1\right)|z|^{\rho-2}\alpha(|z|^2)\right\}$$
$$\cdot\,|\langle z, \bar{\zeta}\rangle|^2 + \left(|z|^\rho\,\alpha'(|z|^2) + \frac{\rho}{2}\,|z|^{\rho-1}\,\alpha(|z|^2)\right)|\zeta|^2.$$

By the Schwarz-Bunjakovskiĭ inequality it follows now that

$$L(\Phi_1; z, \zeta) \geqq \alpha''(|z|^2)|z|^\rho|\langle z, \bar{\zeta}\rangle|^2$$
$$+ \left\{(\rho + 1)|z|^{\rho-1}\,\alpha'(|z|^2) + \frac{\rho^2}{4}\,|z|^{\rho-2}\,\alpha(|z|^2)\right\}|z|^2|\zeta|^2.$$

Hence by (3.6.61) it follows that $L(\Phi_1; z, \zeta) \geqq 0$ $\forall\, \zeta \in C^n\backslash\{0\}$, and thus the function $\Phi_1(|z|)$ is plurisubharmonic in $C^n\backslash\{0\}$. Note that (3.6.61) implies that $\Phi_1'(r) = 2r\alpha'(r^2)r^\rho + \rho\alpha(r^2)r^\rho \geqq 0$. Therefore the function $\Phi_1(r)$ is monotone, and hence $\Phi_1(|z|)$ is plurisubharmonic in the entire space C^n.

[40] However, $\Phi_1(r)$ need not be convex.

Now consider the convolution of $\Phi_1(|z|)$ and the function $\eta(z)$ of Lemma 3.6.5. It is clear that this convolution is a function of $|z|$, and we have thus defined a function $\Phi_2(r)$ such that $\Phi_2(|z|) = \Phi_1(|z|) * \eta(z)$. Unlike the function $\Phi_1(|z|)$, which is generally infinitely differentiable only in $C^n \setminus \{0\}$, the function $\Phi_2(|z|)$ is infinitely differentiable throughout C^n. It is also obvious that $\Phi_2(r)$ satisfies all the requirements of the lemma, except perhaps that $\Phi_2(|z|)$ need not be strictly plurisubharmonic. The required function is therefore

$$\Phi(r) = \Phi_2(r) + \lambda \ln(1 + r^2), \qquad \lambda > 0.$$

Q.E.D.

Before proceeding to the next lemma, we first introduce some notation. Let $C^{(j)}$ denote the subspace of C^n consisting of the points $z^j = (z_1, \cdots, z_j, 0, \cdots, 0)$, and dV^j the volume element of the space $C^{(j)}$ in the metric of C^n. Further, given a function $\psi(z)$, set $\psi_0(z) = \psi(z)$, and let ψ_j', $j = 1, \cdots, n$, denote the regularization of

$$\sup_{|\zeta_j| < 2} \psi_{j-1}(z_1, \cdots, z_{j-1}, z_j + \zeta_j, z_{j+1}, \cdots, z_n),$$

and ψ_j the convolution $\psi_j' * \eta$, where $\eta(z)$ is the same function as in Lemma 3.6.5. It is assumed here that $\eta(z)$ is the same for all $j = 1, \cdots, n$.

LEMMA 3.6.8. *Let the function $\psi(z)$ be plurisubharmonic in C^n, and $f(z^{j-1})$ holomorphic on $C^{(j-1)}$ and such that*

$$\int_{C^{(j-1)}} |f|^2 \exp\{-\psi_{j-1}(z^{j-1})\}(1 + |z^{j-1}|^2)^{-3(j-1)+1}\, dV^{j-1} < \infty. \quad (3.6.61')$$

Then there exists a function $g(z^j)$, holomorphic on $C^{(j)}$, such that $g(z^{j-1}) = f(z^{j-1})$ and

$$\int_{C^{(j)}} |g|^2 e^{-\psi_j(z^j)}(1 + |z^j|^2)^{-3j+1}\, dV^j < \infty. \quad (3.6.62)$$

PROOF. [41] Define a function $\omega_1(t)$ of the real variable t on $[0, \infty)$ by $\omega_1(t) = 1$ for $t < 1/2$, $\omega_1(t) = 0$ for $t > 1$, and $\omega_1(t) = 2(1 - t)$ for $t \in [1/2, 1]$. Smooth this function over the intervals $(1/2 - \varepsilon, 1/2 + \varepsilon)$ and $(1 - \varepsilon, 1 + \varepsilon)$, $0 < \varepsilon < 1/4$, by circular arcs, and denote the resulting function by $\omega_2(t)$. Now set $\omega(z_j) = \omega_2(|z_j|)$. It is clear that $|\partial\omega/\partial\bar{z}_j| \leqq 1$ for all $z_j \in C^1$. Let us look for the function $g(z^j)$ in the form

$$g(z^j) = \omega(z_j) f(z^{j-1}) - z_j X(z^j).$$

For any function $X(z^j)$ this function satisfies the condition $f(z^{j-1}) = g(z^{j-1})$. It will be holomorphic if $\bar{\partial} g(z^j) = 0$, i.e. if

$$\bar{\partial} X = z_j^{-1} f(z^{j-1}) \frac{\partial \omega}{\partial \bar{z}_j}\, d\bar{z}_j. \quad (3.6.63)$$

[41] The proof of this lemma essentially duplicates that of Theorem 3.6.3.

Denote

$$\alpha = z_j^{-1} f(z^{j-1}) \frac{\partial \omega}{\partial \bar{z}_j} d\bar{z}_j.$$

A direct check shows that $\bar{\partial}\alpha = 0$. Now observe that for $|z_j| < 2$

$$|f(z^{j-1})| \exp(-\psi_j(z^j)) \leqq |f(z^{j-1})| \exp(-\psi^{j-1}(z^{j-1})).$$

It therefore follows from (3.6.61′) that

$$\int_{C^{(j)}} |\alpha|^2 e^{-\psi_j(z^j)} (1 + |z^j|^2)^{-3(j-1)+1} dV^j$$

$$= \int_{|z_j|<1+\varepsilon} \left| \frac{f(z^{j-1})}{z_j} \frac{\partial \omega}{\partial \bar{z}_j} \right|^2 e^{-\psi_j(z^j)} (1 + |z^{(j)}|^2)^{-3(j-1)+1} dV^j$$

$$\leqq 16 \int_{|z_j|<1+\varepsilon} \frac{i}{2} dz_j \wedge d\bar{z}_j \int_{C^{(j-1)}} |f(z^{j-1})|^2 \exp(-\psi_{j-1}(z^{j-1}))(1 + |z^{j-1}|^2)^{-3(j-1)+1} dV^{j-1}$$

$$= 16\pi(1 + \varepsilon)^2 \int_{C^{(j-1)}} |f(z^{j-1})|^2 \exp(-\psi_{j-1}(z^{j-1}))(1 + |z^{j-1}|^2)^{-3(j-1)+1} dV^{j-1} < \infty.$$

Thus, we may apply Theorem 3.6.2, according to which there exists a function $X(z^j)$ satisfying equation (3.6.63) such that

$$\int_{C^{(j)}} |X|^2 e^{-\psi_j(z^j)} (1 + |z^j|^2)^{-3j+2} dV^j$$

$$\leqq 8\pi(1 + \varepsilon)^2 \int_{C^{(j-1)}} |f|^2 e^{-\psi_j(z^{j-1})} (1 + |z^{j-1}|^2)^{-3(j-1)+1} dV^{j-1}.$$

Hence it immediately follows that the entire function $g(z^j)$ defined as above in terms of $X(z^j)$ satisfies the condition

$$\int_{C^{(j)}} |g(z^j)|^2 e^{-\psi_j(z^j)} (1 + |z^j|^2)^{-3j+1} dV^j$$

$$\leqq \text{const} \cdot \int_{C^{(j-1)}} |f|^2 e^{-\psi_{j-1}(z^{j-1})} (1 + |z^{j-1}|^2)^{-3(j-1)+1} dV^{j-1} < \infty.$$

Q.E.D.

Now let $\varphi(z)$ be the function figuring in the condition of Theorem 3.6.6, and let $\varphi_\eta = \varphi * \eta$, where $\eta(z)$ is as in Lemma 3.6.5. We shall assume that the point z^0 lies in the subspace $C^{(1)}$; this obviously involves no loss of generality. It is known (see, for example, Levin [1]) that in the one-variable case an entire function with given indicator exists. Hence there exists an entire function $f(z_1)$ such that $L_r(z_1; f) = \varphi(z^1)$. Set

$$\gamma(r) = \sup_{|z_1|=1} \{\max[r^{-\rho} \ln |f(rz_1)| - \varphi(z^1), 0]\}$$

and define a function $\Phi(z)$ in terms of $\gamma(r)$ as in Lemma 3.6.7. Consider the function

$$\psi = 2(\varphi_\eta + \Phi + \ln(1 + |z|^2)).$$

This function is infinitely differentiable, plurisubharmonic, and such that

$$L_r(z;\psi) = 2L_r(z;\varphi_\eta) = 2\varphi(z),$$

and, moreover,

$$\int_{C^{(1)}} |f(z_1)|^2 \exp(-\psi(z^1) + \ln(1 + |z^1|^2))\, dV^1 < \infty. \tag{3.6.64}$$

Applying Lemma 3.6.8 $n-1$ times (this is possible by (3.6.64)), we get an entire function $g(z)$, $z \in C^n$, such that

$$g(z^1) = f(z_1) \qquad \forall z^1 \in C^{(1)} \tag{3.6.65}$$

and

$$\int_{C^n} |g|^2 e^{-\psi_n(z)} (1 + |z|^2)^{-3n+1}\, dV^n < \infty. \tag{3.6.66}$$

By Lemmas 3.6.6, 3.6.5 and 3.6.4 it follows from (3.6.66) that

$$L_r^*(z;g) \leqq L_r^*(z;\tfrac{1}{2}\psi_n) = \varphi(z) \qquad \forall z \in C^n.$$

On the other hand, by virtue of (3.6.65) we have

$$L_r(z^0;g) = L_r(z^0;f) = \varphi(z^0).$$

Consequently $L_r^*(z^0;g) = \varphi(z^0)$. Q.E.D.

Theorem 3.6.6 will play an essential role in the proof of Theorem 3.5.1. We shall also need the following lemmas.

LEMMA 3.6.9. *Let $u(x)$ be upper semicontinuous and bounded above in a domain $G \Subset R^n$, and let B_u be the set of all functions upper semicontinuous in G and majorized by the function $u(x)$ in G. Then there exists a countable family of functions $u_j \in B_u$, $u_j \not\equiv u$, such that for any function $v(x) \in B_u$ not identically equal to $u(x)$ there exists a function $u_{j_0}(x)$ satisfying the condition $v(x) \leqq u_{j_0}(x)$ for all $x \in G$.*

PROOF. Consider some finite ε-covering of the domain G by closed balls $\bar{E}_\varepsilon$. In each of these balls choose a point x' such that

$$\sup_{E_\varepsilon \ni x'} u(x) \leqq u(x') + \varepsilon.$$

Such sets of points x' can be constructed for an arbitrary sequence of numbers ε_m such that $\varepsilon_m \to 0$ as $m \to \infty$. We then get a countable set M, dense in G, such that

$$u(x) = \overline{\lim_{\substack{x' \to x \\ x' \in M}}} u(x') \qquad \forall x \in G.$$

Each function $v(x) \in B_u$ will be identical with $u(x)$ if $v(x) = u(x)$ for all x in M. Indeed, by the semicontinuity of $v(x)$,

$$v(x) \geqq \overline{\lim_{x' \to x}} v(x') \geqq \overline{\lim_{\substack{x' \to x \\ x' \in M}}} v(x') = \overline{\lim_{\substack{x' \to x \\ x' \in M}}} u(x') = u(x).$$

But $v(x) \leqq u(x)$, and so $u(x) = v(x)$.

Now consider a sequence of points $x^{(j)}$ such that for any $x' \in M$ the equality $x^{(j)} = x'$ holds for infinitely many indices j. Set

$$\chi_j(x) = \begin{cases} u(x^{(j)}) - \varepsilon_j & \forall x \in E_{\varepsilon_j}(x^{(j)}), \\ \sup\limits_{x \in G} u(x) & \forall x \in G \setminus E_{\varepsilon_j}(x^{(j)}), \end{cases}$$

where, as usual, $E_{\varepsilon_j}(x^{(j)}) = \{x: |x - x^{(j)}| < \varepsilon_j\}$. These functions are obviously upper semicontinuous. Since the greatest lower bound of any family of upper semicontinuous functions is upper semicontinuous, the functions

$$u_j(x) = \min\{\chi_j(x), u(x)\}$$

are also upper semicontinuous in G.

We claim that the functions $u_j(x)$ satisfy the required conditions. Indeed, let $v(x) \in B_u$ and $v(x_0) < u(x_0)$ at some point $x_0 \in G$. Then in some δ-neighborhood $E_\delta(x_0)$ of x_0 we have $v(x) < u(x_0) - \eta$, where $\eta = \frac{1}{2}(u(x_0) - v(x_0))$. Let $x' \in M$ be a point in the ball $E_{\delta/2}(x_0)$ such that $u(x') > u(x_0) - \eta/2$. It is clear that such a point always exists. Now let j_0 be such that $x^{(j_0)} = x'$, $\varepsilon_{j_0} < \delta/2$ and $\varepsilon_{j_0} < \eta/2$. Then it is readily seen that $v(x) \leqq u_{j_0}(x)$ for all $x \in G$. This completes the proof.

LEMMA 3.6.10. *Let $\varphi(z)$ be a positively homogeneous plurisubharmonic function in C^n of degree $\rho > 0$. Let B'_φ be the family of all positively homogeneous functions in C^n of degree ρ majorized by the function $\varphi(z)$. Then there exists a countable family of functions $v_j \in B'_\varphi$, $v_j \not\equiv \varphi$, such that for any function $v(z) \in B'_\varphi$ not identically equal to $\varphi(z)$ there exists a function $v_{j_0}(z)$ satisfying the condition $v(z) \leqq v_{j_0}(z)$ for all $z \in G$.*

PROOF. By Lemma 3.6.9 there exists a countable family of functions $u_j(z)$, upper semicontinuous in the ball $|z| < 1$, such that for every function $v \in B'_\varphi$, $v \not\equiv \varphi$, there exists a function u_{j_0} satisfying the condition $v(z) \leqq u_{j_0}(z)$ $\forall z \in \{z: |z| < 1\}$.

Consider the function

$$v_j(z) = \sup_{v \in B''_j} v(z),$$

where

$$B''_j = \{v(z): v \in B'_\varphi, v(z) \leqq u_j(z) \quad \forall z \in \{z: |z| < 1\}\}.$$

The regularization v_j^* of this function is clearly plurisubharmonic in the ball $|z| < 1$, and continuable to the entire space C^n as a positively homogeneous plurisubharmonic function of degree ρ. It is also clear that $v_j^*(z) \leqq u_j(z)$ for $|z| < 1$. Consequently $v_j \equiv v_j^*$.

In view of the properties of the family $u_j(z)$, it follows that the functions $v_j(z)$ constitute the required family. Q.E.D.

Now let $\varphi(z)$ be a positively homogeneous plurisubharmonic function of degree ρ, as in the assumptions of Theorem 3.5.1. Express it as a limit $\lim_{j\to\infty} \varphi_j(z)$, where the positively homogeneous plurisubharmonic functions $\varphi_j(z)$ are infinitely differentiable in $C^n\backslash\{0\}$ and $\varphi_j(z) \geqq \varphi_{j+1}(z)$ $\forall z \in C^n$. That this is possible was proved in § 5. Let E_φ denote the space of all entire functions $f(z)$ such that $L_r^*(z;f) \leqq \varphi(z)$. Topologize this space by the countable family of norms

$$\|f\|_\varphi^{(p)} = \max_{C^n} |f(z)| \exp(-\varphi_p(z) - |z|^\rho/p).$$

It is clear that the space E_φ is complete in this topology. Moreover, as a countably normed space, it is a Fréchet space.[42)]

As done here for the space E_φ, we can now define spaces E_ψ for arbitrary positively homogeneous plurisubharmonic functions $\psi(z)$ of degree ρ. It is obvious that $E_\psi \subset E_\varphi$ if $\psi(z) \leqq \varphi(z)$ for all $z \in C^n$, and moreover $E_\psi \neq E_\varphi$ if also $\psi(z^0) < \varphi(z^0)$ at some point z^0. This follows from Theorem 3.6.6. It is also obvious that the embedding $E_\psi \subset E_\varphi$ is continuous. Hence, as is well known (see Banach [1]), it follows that when $\psi(z^0) < \varphi(z^0)$ the set E_ψ is of the first category in the space E_φ. Consider the countable family of spaces $\{E_{v_j}\}$, where the functions v_j form the family constructed from $\varphi(z)$ in Lemma 3.6.10. Any entire function $f(z) \in E_\varphi$ such that $L_r^*(z^0;f) < \varphi(z^0)$ for some z^0 is in one of the spaces E_{v_j}. Hence it follows immediately that the set E_φ' of all entire functions $f \in E_\varphi$ such that $L_r^*(z;f) = \varphi(z)$ can be expressed as

$$E_\varphi' = E_\varphi \Big\backslash \Big\{ \bigcup_{j=1}^{\infty} E_{v_j} \Big\}.$$

As stated, every set E_{v_j} is of the first category in E_φ. Any countable union of such sets is again a set of the first category, and therefore its complement, i.e. the set E_φ', is dense in E_φ, and a fortiori not empty. Thus there exists an entire function $f(z)$ of order ρ whose indicator $L_r^*(f;z)$ is identical with the given function $\varphi(z)$. This completes the proof of Theorem 3.5.1.

[42)] See, for example, Gel'fand and Šilov [2].

CHAPTER 4

DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

In this chapter we introduce some quantitative characteristics for the distribution of roots of the equation $f(z) = 0$, where $f(z)$ is an entire function in C^n. Entire functions will be constructed which have in some sense minimal growth for a given manifold of zeros. These functions are natural analogs of Weierstrass products.

§ 1. Integration over an analytic set[1]

1. *Manifolds.*[2] A Hausdorff space X is called an *n-dimensional manifold* if each of its points P has a neighborhood V_P homeomorphic to an open ball[3] in R^n.

Thus for some neighborhood of any point P_0 in a manifold X there exist n functions $x_1 = f_1(P), \cdots, x_n = f_n(P)$ which realize a homeomorphism $\alpha_{P_0} : V_{P_0} \to \{x : |x| < 1\}$. The pair (V_{P_0}, α_{P_0}) is called a *chart* of the manifold X, the neighborhood V_{P_0} being the *domain of the chart* and the functions $x_i = f_i(P)$, $i = 1, \cdots, n$, a *system of local coordinates of the chart* (V_{P_0}, α_{P_0}).

Apart from the neighborhoods and homeomorphisms specified in the definition of a manifold, each point of the manifold clearly has other neighborhoods and corresponding homeomorphisms into an open ball of R^n. Any such pair, neighborhood plus corresponding homeomorphism, is also called a chart of the manifold.

Consider any two charts (V, α) and (V', α') and the corresponding coordinate systems $x_i = f_i(P)$ and $y_i = \varphi_i(P)$, $i = 1, \cdots, n$. The charts (V, α) and (V', α') are said to be *C^∞-connected* if $V \cap V' = \varnothing$ or if for $V \cap V' \neq \varnothing$ the functions $x_i = x_i(y_1, \cdots, y_n)$, $i = 1, \cdots, n$, realizing the homeomorphism $\alpha\alpha'^{-1} : \alpha'(V \cap V') \to \alpha(V \cap V')$, are infinitely differentiable.

A manifold X is a *manifold with infinitely differentiable structure* (*C^∞-structure*) if it is provided with a set Π of charts (V, α) such that

(1) $\bigcup_{(v,\alpha)\in\Pi} V = X$, and

(2) any two charts in Π are C^∞-connected.

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[1] This section is auxiliary in nature.

[2] For details on this subject, see, for example, de Rham [1], Bishop and Crittenden [1], or Chern [1].

[3] The ball may be replaced in this definition by any set homeomorphic to it, such as an open simplex, n-dimensional open cube, or the entire space R^n.

The set Π of this definition is called a *basis for the C^∞-structure*. The maximal set of pairwise C^∞-connected charts containing Π is called a *C^∞-structure* or (complete) *atlas* of the manifold.

In the sequel, a manifold endowed with a C^∞-structure will be called a differentiable manifold. Any reference to the charts of a differentiable manifold will be to the charts of its atlas.

Complex or *analytic manifolds* are defined in a similar manner. A $2n$-dimensional manifold X is called a complex or analytic manifold of (complex) dimension n if it is provided with a basis for analytic structure, i.e. a set Π of neighborhoods V and homeomorphisms α of these neighborhoods into an open ball in C^n such that

(1) $\bigcup_{(v,\alpha)\in\Pi} V = X$, and

(2) for any two charts (V, α) and (V', α') of Π such that $V \cap V' \neq \varnothing$, the functions $z_j = z_j(\zeta)$, $\zeta \in C^n$, $j = 1,\cdots,n$, realizing the homeomorphism $\alpha\alpha'^{-1}:\alpha'(V \cap V') \to \alpha(V \cap V')$ are holomorphic.

Before proceeding to examples, we stipulate that throughout the sequel the statement "a topological space X is a differentiable (analytic) manifold" will be understood as the statement that a structure of the above type has been introduced in X in a natural manner.[4)]

EXAMPLE 1. The space R^n is obviously a differentiable manifold, and C^n an analytic manifold.

EXAMPLE 2. A circle $\{w:|w| = r\}$ is a one-dimensional differentiable manifold.

Indeed, the neighborhood $V_{\theta_0} = \{w: w = re^{i\theta}, \theta_0 - \pi < \theta < \theta_0 + \pi\}$ of the point $w = re^{i\theta_0}$, where $\theta_0 \in [0, 2\pi]$, is mapped onto the interval $(-\pi, \pi)$ by the homeomorphism $\alpha_{\theta_0}:re^{i\theta} \to t_{\theta_0} = \theta - \theta^0$. The connection between local coordinates t_{θ_0} and t_{θ_1} is given by $t_{\theta_1} = t_{\theta_0} + \theta_0 - \theta_1$.

EXAMPLE 3. The distinguished boundary of a circular polycylinder in C^n is an n-dimensional infinitely differentiable manifold.

This follows in an obvious manner from the fact that the distinguished boundary of a circular polycylinder in C^n is the cartesian product of circles.

EXAMPLE 4. *Projective space* P_{n-1} is the space whose points are complex "rays" κ, i.e. complex one-dimensional analytic planes

$$\kappa = \{z;\ z \in C^n, z = \lambda w, w \in C^1\},$$

where the fixed point $\lambda \in C^n\backslash\{0\}$defines the ray κ.

Note that, with the sole exception of the origin, each point $z \in C^n$ is in exactly one ray $\kappa \in P_{n-1}$. We define the canonical mapping of $C^n\backslash\{0\}$ onto P_{n-1} as the mapping π taking every point $z \in C^n\backslash\{0\}$ onto the ray κ containing it. The space P_{n-1} is topologized as follows. A set $\omega \subset P_{n-1}$ is said to be a neighborhood of a point $\kappa \in \omega$ if $\omega = \pi V$, where V is a neighborhood in $C^n\backslash\{0\}$ of a point z such that $\pi z = \kappa$.

It is easy to see that P_{n-1} is an $(n-1)$-dimensional complex manifold. Indeed,

[4)] The sense in which this structure is "natural" will be obvious in each individual case.

consider the sets

$$\omega_i = \pi\{z: z_i \neq 0\}, \qquad i = 1,\cdots, n.$$

Each such set is obviously the neighborhood of any of its points, and moreover $\omega_1 \cup \cdots \cup \omega_n = P_{n-1}$. Each ray $\kappa \in \omega_i$ cuts the subspace $\{z: z \in C^n, z_i = 1\}$ in exactly one point. We can therefore introduce local coordinates in ω_i in a natural way, associating each $\kappa \in \omega_i$ with a point $(\zeta_1^{(i)}, \cdots, \zeta_{n-1}^{(i)})$ such that

$$\pi(\zeta_1^{(i)},\cdots, \zeta_{i-1}^{(i)}, 1, \zeta_i^{(i)},\cdots, \zeta_{n-1}^{(i)}) = \kappa.$$

To go from local coordinates $\zeta^{(i)}$ to local coordinates $\zeta^{(j)}$ in the intersection $\omega_i \cap \omega_j$, $i < j$, we use the following equations:

$$\begin{aligned}
&\zeta_l^{(i)} = \zeta_l^{(j)}/\zeta_i^{(j)}, && l = 1,\cdots, i-1,\\
&\zeta_l^{(i)} = \zeta_{l+1}^{(j)}/\zeta_i^{(j)}, && l = i,\cdots, j-2,\\
&\zeta_{j-1}^{(i)} = 1/\zeta_i^{(j)},\\
&\zeta_l^{(i)} = \zeta_l^{(i)}/\zeta_i^{(j)}, && l = j,\cdots, n-1.
\end{aligned}$$

At no point of the intersection $\omega_i \cap \omega_j$ do the local coordinates $\zeta_j^{(i)}$ and $\zeta_i^{(j)}$ vanish. The coordinates $\zeta_1^{(i)}$ $(\zeta_1^{(j)})$ are therefore holomorphic functions of the variables $\zeta_1^{(j)}$ $(\zeta_1^{(i)})$, and so P_{n-1} is an $(n-1)$-dimensional complex manifold.

2. *Orientation of a manifold.* A differentiable manifold X is said to be *orientable* if there exists a set of charts (U_j, β_j) such that $\bigcup U_j = X$ and when $U_j \cap U_i \neq \varnothing$ the passage in $U_j \cap U_i$ from local coordinates of the chart (U_j, β_j) to those of the chart (U_i, β_i) is realized by a system of functions with positive Jacobian. Let $\Pi^+(X)$ $(\Pi^-(X))$ denote the set of charts (V_P, α_P) such that if $V_P \cap U_j \neq \varnothing$ the passage in $V_P \cap U_j$ from local coordinates of (V_P, α_P) to those of (U_j, β_j) is realized by a system of functions with positive (negative) Jacobian. It is readily seen that, for any two charts whose domains have nonempty intersection, the passage from the local coordinates of the one to those of the other is realized by a system of functions with positive Jacobian if both charts belong to the same set $\Pi^+(X)$ or $\Pi^-(X)$, and with negative Jacobian otherwise. Let us assign the set $\Pi^+(X)$ the number $\varepsilon(\Pi^+) = 1$ and the set $\Pi^-(X)$ the number $\varepsilon(\Pi^-) = -1$. These numbers are called the *orientations* of the manifold X, and the manifold together with its orientations is said to be *oriented.* Thus, to orient an orientable manifold, we must partition the atlas into sets $\Pi^+(X)$ and $\Pi^-(X)$ and then define the number ε_X, equal to $+1$ for the charts in $\Pi^+(X)$ and to -1 for the charts in $\Pi^-(X)$.

Let X be an n-dimensional oriented manifold and G a domain in X whose boundary ∂G is an $(n-1)$-dimensional manifold in the topology induced by that of X. Let P be a point of ∂G, (V_P, α_P) and (V_P^*, α_P^*) charts of the manifolds X and ∂G, respectively, and $x_1,\cdots, x_n$ and $t_1,\cdots, t_{n-1}$ local coordinates in these charts. Without loss of generality, we may assume that the point P corresponds to the origin in both local systems. The homeomorphisms α_P and α_P^* generate a ho-

meomorphism

$$\alpha_P\alpha_P^{*-1}\colon \alpha_P^*(V_P \cap V_P^*) \to \alpha_P(V_P \cap V_P^*),$$

which is realized by a certain system of functions

$$x_1 = f_1(t_1, \cdots, t_{n-1}), \cdots, x_n = f_n(t_1, \cdots, t_{n-1}). \tag{4.1.1}$$

If any such system (4.1.1) consists of continuously differentiable functions and the rank of its Jacobian is $n-1$, the domain G is said to be a domain with smooth boundary.

Let G be a domain with smooth boundary ∂G on a manifold X, and let (V_P, α_P), (V_P^*, α_P^*), $x_1, \cdots, x_n, t_1, \cdots, t_{n-1}$ and $f_1, \cdots, f_n$ be as before. Since the rank of the Jacobian of (4.1.1) is $n-1$, the surface $\alpha_P(V_P \cap V_P^*)$ defined by these functions has a tangent plane and a normal at each of its points. Set

$$A_f^{(j)} = (-1)^{j+n} \frac{\partial(f_1, \cdots, f_{j-1}, f_{j+1}, \cdots, f_n)}{\partial(t_1, \cdots, t_{n-1})}$$[5]

and consider the normals to $\alpha_P(V_P \cap V_P^*)$ in the neighborhood of the origin; these are obviously defined by the equalities

$$x_i = f_i(t_1, \cdots, t_{n-1}) + A_f^{(i)}(t_n), \qquad i = 1, \cdots, n, \tag{4.1.2}$$

with fixed $t_1, \cdots, t_{n-1}$ and variable t_n, $-\infty < t_n < \infty$. Note that for this system of functions

$$\left.\frac{\partial(x_1, \cdots, x_n)}{\partial(t_1, \cdots, t_n)}\right|_{t_n=0} = \sum_{j=1}^{n} (A_f^{(j)})^2 > 0.$$

Consequently the equalities (4.1.2) realize a homeomorphism of some neighborhood of the origin in $R_{(x)}^n$ onto a corresponding neighborhood of the origin in $R_{(t)}^n$. Under this homeomorphism, the surface $\alpha_P(V_P \cap V_P^*)$ is mapped into a region of the coordinate plane $\{t: t_n = 0\}$ containing the origin. Denote this homeomorphism by β_f, and consider the corresponding orientation of the manifold ∂G. That is to say, if $(V_P, \alpha_P) \in \Pi^+(X)$ $((V_P, \alpha_P) \in \Pi^-(X))$, the chart (V_P^*, α_P^*) is assigned to the set $\Pi^+(\partial G)$ $(\Pi^-(\partial G))$ if, for some sufficiently small $\varepsilon > 0$,

$$\beta_f(\alpha_P(G \cap V_P) \cap \{x: |x| < \varepsilon\}) \subset \{t: t_n < 0\}.$$

Similarly, if

$$\beta_f(\alpha_P(G \cap V_P) \cap \{x: |x| < \varepsilon\}) \subset \{t: t_n > 0\},$$

the chart (V_P^*, α_P^*) is assigned to the set $\Pi^+(\partial G)$ if $(V_P, \alpha_P) \in \Pi^-(X)$, and to the set $\Pi^-(\partial G)$ if $(V_P, \alpha_P) \in \Pi^+(X)$.

It can be shown that the sets $\Pi^{\pm}(\partial G)$ are well defined, i.e. any chart (V_P^*, α_P^*) is

[5] As usual, $\partial(f_1, \cdots, f_{j-1}, f_{j+1}, \cdots, f_n)/\partial(t_1, \cdots, t_{n-1})$ denotes the Jacobian of the indicated system of functions.

asssigned to one of the sets $\Pi^{\pm}(\partial G)$ irrespective of the choice of the chart (V_P, α_P); and that the Jacobian of the passage from local coordinates in $(V_P^*, \alpha_P^*) \in \Pi^+(\partial G)$ to local coordinates in any chart $(U^*, \beta^*) \in \Pi^+(\partial G)$ such that $P \in U^*$ is positive in $V_P^* \cap U^*$.

3. *Volume element of a manifold.* In the space R^m, consider the l-dimensional surface $(l < m)$

$$M = \{x: x = x(t), t \in R^l, |t| < 1\},$$

where $x(t)$ is a continuously differentiable vector-valued function with Jacobian of rank l. The standard metric in R^m induces a metric on M:

$$ds^2 = \sum_{i,j=1}^{l} G_{i,j}\, dt_i\, dt_j,$$

where, as is readily seen,

$$G_{i,j} = \langle \partial x/\partial t_i, \partial x/\partial t_j \rangle.$$

Under a transformation of variables $t = t(s)$, $s \in R^l$, we obviously have

$$ds^2 = \sum_{i,j=1}^{l} G_{i,j}^*\, ds_i\, ds_j,$$

where the functions $G_{i,j}^*$ are related to the functions $G_{i,j}$ by

$$G_{i,j}^* = \sum_{p,q=1}^{l} G_{p,q} \frac{\partial t_p}{\partial s_i} \frac{\partial t_q}{\partial s_j}. \tag{4.1.3}$$

It is well known that the (l-dimensional) volume of any l-dimensional region M_1 of the surface M is given by the integral

$$\int_{M_1^*} \sqrt{\det\{G_{i,j}\}}\, dt_1 \wedge \cdots \wedge dt_l.$$

Here M_1^* is the preimage of the set M_1 under the mapping $t \to x(t)$. The odd differential form under the integral sign is called the *volume element of the surface* M and denoted by dV_l.

Now let M be a smooth l-dimensional oriented manifold, with a Riemann metric defined in local coordinates by

$$ds^2 = \sum_{i,j=1}^{l} G_{i,j}\, dt_i\, dt_j. \tag{4.1.4}$$

Under a transformation of coordinates, the functions $G_{i,j}$ of the local coordinates transform according to formulas (4.1.3), and they are such that the form ds^2 is positive definite throughout M. By analogy with the preceding case, we define the *volume element of the manifold* M by

$$dV_l = \sqrt{\det\{G_{i,j}\}}\, dt_1 \wedge \cdots \wedge dt_l;$$

and, accordingly, the volume of a domain $D \subset M$ is defined by the integral $\int_D dV_l$. We are particularly interested in the case of a manifold M of even dimension which admits a complex structure, or, more precisely, which is a complex manifold with Hermitian metric, i.e. a metric defined in local coordinates by a positive definite Hermitian form

$$ds^2 = \sum_{i,j=1}^{l} h_{i,j}\, dz_i\, d\bar{z}_j, \tag{4.1.5}$$

where $h_{i,j} = \bar{h}_{j,i}$ and, under any passage to new coordinates $\zeta_1, \cdots, \zeta_l$, the coefficients $h_{i,j}$ transform according to

$$h^*_{i,j} = \sum_{p,q=1}^{l} h_{p,q} \frac{\partial z_p}{\partial \zeta_i} \frac{\partial \bar{z}_q}{\partial \bar{\zeta}_j}.$$

Going over to real coordinates

$$t_{2j-1} = (z_j + \bar{z}_j)/2, \qquad t_{2j} = (z_j - \bar{z}_j)/2i,$$

we transform (4.1.5) to

$$ds^2 = \sum_{p,q=1}^{2l} G_{p,q}\, dt_p\, dt_q,$$

where, as is shown by a direct calculation,

$$G_{2\nu-1,2\mu-1} = \frac{h_{\nu,\mu} + h_{\mu,\nu}}{2}, \qquad G_{2\nu,\,2\mu} = \frac{h_{\nu,\mu} + h_{\mu,\nu}}{2},$$

$$G_{2\nu-1,\,2\mu} = \frac{h_{\nu,\mu} - h_{\mu,\nu}}{2i}, \qquad G_{2\nu,2\mu-1} = \frac{h_{\mu,\nu} - h_{\nu,\mu}}{2i}.$$

Thus the matrix $\{G_{i,j}\}$ is built up from blocks

$$\begin{pmatrix} \dfrac{h_{\nu,\mu} + h_{\mu,\nu}}{2} & \dfrac{h_{\nu,\mu} - h_{\mu,\nu}}{2i} \\ \dfrac{h_{\mu,\nu} - h_{\nu,\mu}}{2i} & \dfrac{h_{\nu,\mu} + h_{\mu,\nu}}{2} \end{pmatrix}.$$

Hence by elementary transformations we see that

$$\det\{G_{i,j}\} = \begin{vmatrix} h_{1,1} \cdots h_{1,l} & & \\ \cdots\cdots & 0 & \\ h_{l,1} \cdots h_{l,l} & & \\ & h_{1,1} \cdots h_{l,1} & \\ 0 & \cdots\cdots & \\ & h_{1,l} \cdots h_{l,l} & \end{vmatrix} = \{\det\{h_{\nu,\mu}\}\}^2.$$

Consequently the volume element dV_{2l} of the manifold is defined in local coordinates by

$$dV_{2l} = \det\{h_{\nu,\mu}\}\, dt_1 \wedge \cdots \wedge dt_{2l},^{6)}$$

or, equivalently,

$$dV_{2l} = (i/2)^l \det\{h_{\nu,\mu}\}\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_l. \tag{4.1.6}$$

Now consider the differential form

$$dV_2 = \frac{i}{2} \sum_{\nu,\mu=1}^{l} h_{\nu,\mu}\, dz_\nu \wedge d\bar{z}_\mu \tag{4.1.7}$$

on the manifold M. Note that (4.1.6) and (4.1.7) do not contradict each other. Indeed, let N be a complex one-dimensional submanifold of M. It is defined by analytic functions $z_i = z_i(w)$, where $w \in C^1$ and z_i are local coordinates in a certain chart of the manifold M. The metric of M induces a Hermitian metric on N:

$$ds^2 = \sum_{\nu,\mu=1}^{l} h_{\nu,\mu}\, dz_\nu\, d\bar{z}_\mu = \left\{ \sum_{\nu,\mu=1}^{l} h_{\nu,\mu} \frac{\partial z_\nu}{\partial w} \frac{\partial \bar{z}_\mu}{\partial \bar{w}} \right\} dw\, d\bar{w},^{7)}$$

whence by (4.1.6) it follows that

$$dV_2 = \frac{i}{2} \sum_{\nu,\mu=1}^{l} h_{\nu,\mu} \frac{\partial z_\nu}{\partial w} \frac{\partial \bar{z}_\mu}{\partial \bar{w}}\, dw \wedge d\bar{w}.$$

On the other hand, on N we have

$$\frac{i}{2} \sum_{\nu,\mu=1}^{l} h_{\nu,\mu}\, dz_\nu \wedge d\bar{z}_\mu = \frac{i}{2} \sum_{\nu,\mu=1}^{l} h_{\nu,\mu} \frac{\partial z_\nu}{\partial w} \frac{\partial \bar{z}_\mu}{\partial \bar{w}}\, dw \wedge d\bar{w} = dV_2.$$

As an example, consider the complex one-dimensional analytic surface $M = \{z\colon z_j = z_j(w), j = 1, \cdots, n, w \in C^1, |w| < 1\}$ in C^n. We have

$$dV_2 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n),$$

and so the area $V_2(M)$ of this surface is defined by

$$V_2(M) = \frac{i}{2} \int_{|w|<1} dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n = \frac{i}{2} \sum_{j=1}^{n} \int_{|w|<1} dz_j \wedge d\bar{z}_j. \tag{4.1.8}$$

The integral

$$\frac{i}{2} \int_{|w|<1} dz_j \wedge d\bar{z}_j$$

is clearly the area of the projection of the surface M on the z_j-plane. Thus equality (4.1.8) means that *the area of an analytic complex one-dimensional surface M is equal to the sum of areas of its projections on the coordinate planes* (see Wirtinger [1]). It can be shown that a similar statement is true for analytic surfaces of arbi-

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6) Since the form (4.1.5) is positive definite, the determinant $\det\{h_{\nu,\mu}\}$ is positive.

7) By the analyticity of the functions $z_i(w)$, we have $z_i(w)\partial z_i/\partial \bar{w} = \partial \bar{z}_i/\partial w = 0$.

trary dimension.

We now consider a complex l-dimensional submanifold N_l of a complex n-dimensional manifold M. It is defined on each chart of M by certain analytic functions $z_i = z_i(w)$, where $w \in C^l$ and z_i are local coordinates in M. The metric of N_l is obviously defined by

$$ds^2 = \sum_{p,q=1}^{l} \left\{ \sum_{\nu,\mu=1}^{n} h_{\nu,\mu} \frac{\partial z_\nu}{\partial w_p} \frac{\partial \bar{z}_\mu}{\partial \bar{w}_q} \right\} dw_p \, d\bar{w}_q,$$

and the volume element by

$$dV_{2l} = (i/2)^l \det \{h^*_{p,q}\} \, dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_l \wedge d\bar{w}_l,$$

where

$$h^*_{p,q} = \sum_{\nu,\mu=1}^{n} h_{\nu,\mu} \frac{\partial z_\nu}{\partial w_p} \frac{\partial \bar{z}_\mu}{\partial \bar{w}_q}.$$

We now prove the following useful relation:

$$dV_{2l} = (dV_2)^l / l!, \tag{4.1.9}$$

where $(dV_2)^l = dV_2 \wedge \cdots \wedge dV_2$ (l factors). Indeed, on N_l we have

$$dV_2 = \frac{i}{2} \sum_{\nu,\mu=1}^{n} h_{\nu,\mu} \, dz_\nu \wedge d\bar{z}_\mu = \frac{i}{2} \sum_{p,q=1}^{l} h^*_{p,q} \, dw_p \wedge d\bar{w}_q.$$

Therefore

$$\begin{aligned}(dV_2)^l &= \left(\frac{i}{2}\right)^l \prod_{j=1}^{l} \left(\sum_{p_j,q_j=1}^{l} h^*_{p_j,q_j} dw_{p_j} \wedge d\bar{w}_{q_j} \right) \\ &= \left(\frac{i}{2}\right)^l \sum_{p_1,q_1,\cdots p_l,q_l=1}^{l} h^*_{p_1,q_1} h^*_{p_2,q_2} \cdots h^*_{p_l,q_l} \, dw_{p_1} \wedge d\bar{w}_{q_1} \wedge \cdots \wedge dw_{p_l} \wedge d\bar{w}_{q_l}.\end{aligned}$$

According to the rules for multiplication of exterior forms, the only possible nonvanishing terms in this sum are those for which $p_1 \neq p_2 \neq \ldots \neq p_l$ and $q_1 \neq q_2 \neq \ldots \neq q_l$. Arranging the differentials dw_p and $d\bar{w}_q$ in these terms in order of increasing indexes, we get

$$\begin{aligned}(dV_2)^l &= \left(\frac{i}{2}\right)^l \sum_{\substack{p_1 \neq p_2 \neq \cdots \neq p_l \\ q_1 \neq q_2 \neq \cdots \neq q_l}} \operatorname{sign} \begin{pmatrix} p_1 \cdots p_l \\ q_1 \cdots q_l \end{pmatrix} h^*_{p_1,q_1} \cdots h^*_{p_l,q_l} \, dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_l \\ &= \left(\frac{i}{2}\right)^l l! \sum_{q_1,\cdots,q_l=1}^{l} \operatorname{sign} \begin{pmatrix} 1 \cdots l \\ q_1 \cdots q_l \end{pmatrix} h^*_{1,q_1} h^*_{2,q_2} \cdots h^*_{l,q_l} \, dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_l \\ &= \left(\frac{i}{2}\right)^l l! \det \{h^*_{p,q}\} \, dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_1 = l! \, dV_{2l}.\end{aligned}$$

This proves (4.1.9).

4. *Metric of a projective space.* As before, let P_{n-1} be complex projective space, i.e. the space whose points are complex rays $\kappa = \{z : z = \lambda w, \ |w| < \infty\}$, where

$\lambda \in C^n$ defines the ray κ. Here λ may be any point $z \in \kappa$, $z \neq 0$. In terms of the canonical projection $\pi: C^n \to P_{n-1}$, the fact that a point z is in the ray κ is expressed as $\pi z = \kappa$. We now make P_{n-1} a metric space, defining the distance $\rho(\kappa, \kappa')$ between points $\kappa \in P_{n-1}$ and $\kappa' \in P_{n-1}$ by

$$\rho(\kappa, \kappa') = \min_{\substack{\pi z = \kappa; \pi z' = \kappa' \\ |z| = |z'| = 1}} |z - z'|.$$

If κ and κ' are rays defined by $\lambda \in C^n$ and $\lambda' \in C^n$, then

$$\begin{aligned} \rho^2(\kappa, \kappa') &= \min_{\substack{|\lambda w| = \lambda' w'| = 1 \\ w \in C^1, w' \in C^1}} |\lambda w - \lambda' w'|^2 = \min_{|\lambda w| = |\lambda' w'| = 1} \langle \lambda w - \lambda' w', \bar{\lambda} \bar{w} - \bar{\lambda}' \bar{w}' \rangle \\ &= \min_{\theta, \varphi} \left\langle \frac{\lambda}{|\lambda|} e^{i\theta} - \frac{\lambda'}{|\lambda'|} e^{i\varphi}, \frac{\bar{\lambda}}{|\lambda|} e^{-i\varphi} - \frac{\bar{\lambda}'}{|\lambda'|} e^{-i\varphi} \right\rangle \\ &= 2 - \max_{\theta, \varphi} \left(e^{i(\theta - \varphi)} \frac{\langle \lambda, \bar{\lambda}' \rangle}{|\lambda| |\lambda'|} + e^{i(\varphi - \theta)} \frac{\langle \bar{\lambda}, \lambda' \rangle}{|\lambda| |\lambda'|} \right) = 2 - 2 \frac{|\langle \lambda, \bar{\lambda}' \rangle|}{|\lambda| |\lambda'|}. \end{aligned} \tag{4.1.10}$$

The squared line element $d\sigma^2$, or, what is the same, the Hermitian metric in P_{n-1}, is obtained by setting $\lambda' = \lambda + d\lambda$ in (4.1.10) and computing the principal part of the expansion of $\rho^2(\kappa, \kappa')$ in powers of $d\lambda_1, \cdots, d\bar{\lambda}_n$. The result is

$$\begin{aligned} \rho^2(\kappa, \kappa') &= 2 - 2 \frac{|\langle \lambda, \bar{\lambda} + d\bar{\lambda} \rangle|}{|\lambda| |\bar{\lambda} + d\bar{\lambda}|} \\ &= 2 - 2 \sqrt{\frac{(\langle \lambda, \bar{\lambda} \rangle + \langle \lambda, d\bar{\lambda} \rangle) \cdot (\langle \lambda, \bar{\lambda} \rangle + \langle \bar{\lambda}, d\lambda \rangle)}{\langle \lambda, \bar{\lambda} \rangle (\langle \lambda, \bar{\lambda} \rangle + \langle \lambda, d\bar{\lambda} \rangle + \langle \bar{\lambda}, d\lambda \rangle + \langle d\lambda, d\bar{\lambda} \rangle)}} \\ &= 2 \left(1 - \sqrt{1 - \frac{\langle \lambda, \bar{\lambda} \rangle \langle d\lambda, d\bar{\lambda} \rangle - \langle \lambda, d\bar{\lambda} \rangle \langle d\lambda, \bar{\lambda} \rangle}{|\lambda|^2 (\langle \lambda, \bar{\lambda} \rangle + \langle \lambda, d\bar{\lambda} \rangle + \langle d\lambda, \bar{\lambda} \rangle + \langle d\lambda, d\bar{\lambda} \rangle)}} \right) \\ &= \frac{\langle \lambda, \bar{\lambda} \rangle \langle d\lambda, d\bar{\lambda} \rangle - \langle \lambda, d\bar{\lambda} \rangle \langle d\lambda, \bar{\lambda} \rangle}{|\lambda|^2 (|\lambda|^2 + \langle \lambda, d\bar{\lambda} \rangle + \langle d\lambda, \bar{\lambda} \rangle + \langle d\lambda, d\bar{\lambda} \rangle)} + o(|d\lambda|^2), \end{aligned}$$

and so

$$d\sigma^2 = \frac{\langle \lambda, \bar{\lambda} \rangle \langle d\lambda, d\bar{\lambda} \rangle - \langle \bar{\lambda}, d\lambda \rangle \langle \lambda, d\bar{\lambda} \rangle}{\langle \lambda, \bar{\lambda} \rangle^2}. \tag{4.1.11}$$

As could be expected, this quadratic form is independent of the choice of the point λ defining the ray κ, i.e. $d\sigma^2$ is invariant under transformations $\lambda \to \lambda w$, $w \in C^1 \setminus \{0\}$. To derive an expression for $d\sigma^2$ in terms of local coordinates on P_{n-1}, it is now sufficient to express the quantities $\lambda_1, \cdots, \lambda_n$ in (4.1.11) in terms of local coordinates and to carry out the operations figuring in (4.1.11).

5. *Volume element in projective space.* By the rules established in § 1.3, the volume element dW_{2n-2} in the projective space P_{n-1} is defined by

$$dW_{2n-2} = \frac{1}{(n-1)!} (dW_2)^{n-1}, \tag{4.1.12}$$

where the exterior differential form dW_2 is constructed as shown previously from the quadratic form $d\sigma^2$:

$$dW_2 = \frac{i}{2} \frac{\langle \lambda, \bar{\lambda} \rangle \sum_1^n d\lambda_j \wedge d\bar{\lambda}_j - \langle \bar{\lambda}, d\lambda \rangle \wedge \langle \lambda, d\bar{\lambda} \rangle}{\langle \lambda, \bar{\lambda} \rangle^2}. \tag{4.1.13}$$

To derive an expression for dW_{2n-2} in a given system of local coordinates, we may proceed in either of two ways: (a) as in §1.4, express $\lambda_1, \cdots, \lambda_n$ in (4.1.13) in terms of local coordinates, or (b) express $d\sigma^2$ in terms of local coordinates and then directly use (4.1.9).

EXAMPLE. Let M be the open set in P_{n-1} formed by all rays $\kappa = \{z : z = \lambda w, |w| < \infty, w \in C^1, \lambda \in C^n\}$ with $\lambda_n = 1$. The space C^{n-1} of the variables $\lambda_1, \cdots, \lambda_{n-1}$ is obviously a local-coordinate space for M. In terms of these coordinates, by (4.1.11) we have

$$d\sigma^2 = \frac{(1 + \sum_{p=1}^{n-1} \lambda_p \bar{\lambda}_p) \sum_{p=1}^{n-1} d\lambda_p \, d\bar{\lambda}_p - \sum_{p,q=1}^{n-1} \bar{\lambda}_p \lambda_q \, d\lambda_p \, d\bar{\lambda}_q}{(1 + \sum_{p=1}^{n-1} \lambda_p \bar{\lambda}_p)^2}. \tag{4.1.14}$$

Consequently

$$dW_{2n-2} = \left(\frac{i}{2}\right)^{n-1} \det \left\{ \frac{\delta_{p,q}}{1 + \sum_{p=1}^{n-1} \lambda_p \bar{\lambda}_p} - \frac{\bar{\lambda}_p \lambda_q}{(1 + \sum_{p=1}^{n-1} \lambda_p \bar{\lambda}_p)^2} \right\},$$

where $\delta_{p,q} = 0$ for $p \neq q$ and $\delta_{p,p} = 1$. Computing the determinant in this expression by means of elementary transformations, we get

$$dW_{2n-2} = \left(\frac{i}{2}\right)^{n-1} \frac{d\lambda_1 \wedge d\bar{\lambda}_1 \wedge \cdots \wedge d\bar{\lambda}_{n-1}}{(1 + \sum_{p=1}^{n-1} \lambda_p \bar{\lambda}_p)^n}. \tag{4.1.15}$$

Denoting the volume element in the space of $\lambda_1, \cdots, \lambda_{n-1}$ by dV_{2n-2}, we can rewrite (4.1.15) in a more compact form:

$$dW_{2n-2} = \frac{dV_{2n-2}}{(1 + \sum_{p=1}^{n-1} |\lambda_p|^2)^n}. \tag{4.1.16}$$

We now observe that the dimension of the manifold $P_{n-1} \setminus M$ is lower than that of the space P_{n-1}. Therefore, if $f(\kappa)$ is a function integrable on P_{n-1}, then

$$\int_{P_{n-1}} f(\kappa) dW_{2n-2} = \int_M f(\kappa) dW_{2n-2}$$

and so

$$\int_{P_{n-1}} f(\kappa) dW_{2n-2} = \int_{C^{n-1}} \frac{f(\kappa(\lambda)) dV_{2n-2}}{(1 + \sum_{p=1}^{n-1} |\lambda_p|^2)^n},$$

where $f(\kappa(\lambda))$ is the function generated by $f(\lambda)$ in our system of local coordinates.

6. *Analytic sets.*[8] Let G be a domain in C^n. A set $S \subset G$ is said to be *analytic*

[8] For details on analytic sets, see, e.g., Hervé [1], Gunning-Rossi [1].

if for every point $z^0 \in S$ there are a neighborhood υ_{z^0} of z^0 and a system of functions $f_i(z)$, $i = 1, \cdots, N$, holomorphic in υ_{z^0}, such that

$$S \cap \upsilon_{z^0} = \{z: z \in \upsilon_{z^0}, f_i(z) = 0, i = 1, \cdots, N\}.$$

In particular, the set $\chi_f = \{z: f(z) = 0\}$, where $f(z)$ is an entire function, is an analytic set.

A point z^0 of an analytic set S is said to be *regular* if it has a neighborhood υ_{z^0} such that the set $S \cap \upsilon_{z^0}$ has the representation

$$S \cap \upsilon_{z^0} = \{z: f_1(z) = \cdots = f_{n-r}(z) = 0\},$$

where the functions $f_1(z), \cdots, f_{n-r}(z)$ are holomorphic in υ_{z^0} and the rank of their Jacobian is $n - r$. It follows from the implicit function theorem that this definition of a regular point is equivalent to the following one.

A point $z^0 \in S$ is *regular* if there exist a neighborhood υ_{z^0} of z^0 and a mapping biholomorphic in υ_{z^0} such that

$$\alpha(S \cap \upsilon_{z^0}) = \alpha(\upsilon_{z^0}) \cap L,$$

where $L \subset C^n$ is an analytic plane of complex dimension r, $0 \leqq r \leqq n$.

The number r figuring in both definitions is called the *dimension of the analytic set at the point,* denoted by $\dim_{z^0} S$.

Denote the set of ordinary points of an analytic set S by S^*. A point $z \in S$ is said to be a *singular point of* S if $z \notin S^*$. The dimension of the set S at a singular point z^0 is defined as

$$\dim_{z^0} S = \lim_{\varepsilon \to 0} \sup_{z \in S^*,\ |z - z^0| < \varepsilon} \dim_z S.$$

Finally, the *dimension of the set* S is defined to be the number

$$\dim S = \sup_{z \in S} \dim_z S.$$

It is readily seen that $\dim_a S$ remains constant on each component of S^*. Moreover, *every component of* S^* *is an analytic manifold of dimension equal to that of* S *at the points of the component.*

It is known that the set $S \setminus S^*$, i.e. the set of singular points, is also an analytic set, with $\dim S \setminus S^* < \dim S$.[9] An inductive application of this statement easily shows that *every analytic set* S *is the union of at most countably many analytic manifolds embedded in* C^n. Moreover, any domain $G' \Subset G$ can intersect only finitely many of these manifolds.

Some additional properties of analytic sets follow.[10]

Let S be a set analytic in the polydisk $\mathscr{E} = \{z: |z_i| < r_i, i = 1, \cdots, n\}$.

(a) *If the intersection* $\mathscr{E} \cap \{z: z_i = z_i^0, i = 1, \cdots, n-1\}$, *where* $|z_i^0| < r_i$, $i = 1, \cdots,$

[9] For details on this theorem, which is due to H. Cartan, see, for example, Hervé [1].

[10] For proofs of these properties, except (c), which is quite easily reduced to the case $n = 1$, see, for example, Hervé [1].

$n-1$, is not a subset of S, then for any $r_n' < r_n$ the set $S \cap \{z: z_i = z_i^0,\ i = 1, \cdots, n-1, |z_n| < r_n'\}$ is finite or empty.

(b) *If $S = \chi_f \cap \mathscr{E} \neq \varnothing$, where $\chi_f = \{z: f(z) = 0\}$, $f(z)$ a holomorphic function, then*

$$\dim S = \dim \chi_f = n - 1.$$

(c) *If $\dim S < n$, then the intersection of S with any distinguished boundary $\mathfrak{S} = \{z: |z_i| = r_i' < r_i,\ i = 1, \cdots, n\}$ has zero Lebesgue measure relative to $\mathfrak{S}$.*

(d) *If $f(z)$ is holomorphic in $\mathscr{E}$ and $S' = \{z: z \in \mathscr{E}, f(z) = 0\}$, then either $\dim_{z^0}(S \cap S') = \dim_{z^0} S - 1$ at every point $z^0 \in S \cap S'$, or S' contains a component of the set S.*

7. *The set χ_f.* Consider the ring $\mathfrak{O}_a^{(n)}$ of functions holomorphic at a point $a \in C^n$. Two functions f and φ in this ring are said to be *equivalent* if the point a has a neighborhood in which $f = \varphi\psi$, where $\psi \in \mathfrak{O}_a^{(n)}$ and $\psi(a) \neq 0$. A function $f \in \mathfrak{O}_a^{(n)}$ is said to be *reducible in the ring* $\mathfrak{O}_a^{(n)}$ (or at the point a) if a has a neighborhood in which $f = \varphi\psi$, where $\psi \in \mathfrak{O}_a^{(n)}$, $\varphi \in \mathfrak{O}_a^{(n)}$, and $\varphi(a) = \psi(a) = 0$. A function $f \in \mathfrak{O}_a^{(n)}$ which is not reducible in $\mathfrak{O}_a^{(n)}$ is said to be *irreducible in* $\mathfrak{O}_a^{(n)}$.

One sufficient condition for reducibility of functions is based on the concept of the multiplicity of a zero. Let $f \in \mathfrak{O}_{z^0}^{(n)}$ and $f(z^0) = 0$. Expand $f(z)$ in a neighborhood of z^0 in powers of $z_1 - z_1^0, \cdots, z_n - z_n^0$:

$$f(z) = \sum_{\|k\|=0}^{\infty} a_k (z - z^0)^k.$$

The *multiplicity* of the zero (root) z^0 is defined as the number

$$\gamma_f(z) = \min_{a_k \neq 0} \|k\|.$$

It is obvious that if $f(z^0) = 0$, then $\gamma_f(z^0) \geqq 1$, and if the function $f(z)$ is reducible at z^0 then $\gamma_f(z^0) \geqq 2$.

Now let z^0 be a *regular* point of the set $\chi_f = \{z: f(z) = 0\}$. Then $\gamma_f(z^0) = m$ *if and only if in some neighborhood of z^0 the function $f(z)$ admits the representation*

$$f(z) = (\psi_{z^0}(z))^m, \tag{4.1.17}$$

where $\psi_{z^0} \in \mathfrak{O}_a^{(n)}$ and $\gamma_{\psi_{z^0}}(z^0) = 1$.

Indeed, it follows from the definition of a regular point that there exists a biholomorphic mapping $\alpha: \zeta \to z$ such that $F(\zeta) = f(z(\zeta))$ vanishes in a neighborhood υ_{ζ^0} of the point $\zeta^0 = \alpha^{-1}(z^0)$ if and only if $\zeta_n = \zeta_n^0$. The function $F(\zeta)$ therefore has the following representation in υ_{ζ^0}:

$$F(\zeta) = (\zeta_n - \zeta_n^0)^{m_1} F_1(\zeta), \tag{4.1.18}$$

where the function $F_1(\zeta)$ is holomorphic in υ_{ζ^0} and satisfies the condition

$$F_1(\zeta_1, \cdots, \zeta_{n-1}, \zeta_n^0) \not\equiv 0. \tag{4.1.19}$$

The set $\chi_{F_1} = \{\zeta: \zeta \in \upsilon_{\zeta^0}, F_1(\zeta) = 0\}$ is either of dimension $n-1$ or empty. Hence, since $F(\zeta) \neq 0$ for $\zeta_n \neq \zeta_n^0$, and in view of (4.1.18), we conclude that $F_1(\zeta^0) \neq 0$. Consequently $m_1 = \gamma_F(\zeta^0)$, and since the multiplicity of a root is invariant under biholomorphic mappings, it follows that $m_1 = \gamma_f(z^0)$. When we return to the variables $z_1, \cdots, z_n$, equality (4.1.18) yields the representation (4.1.17) with

$$\psi(z) = (\zeta_n(z) - \zeta_n^0)(F_1(\zeta(z)))^{1/m_1}.$$

It follows from the proposition just proved that $\gamma_f(z)$ *is constant on every component of the set* χ_f^* *of regular points of the set* χ_f.

It is well known[11] that, if the function $f(z)$ is holomorphic in some polydisk $\mathcal{E}(a, r)$, then every component of χ_f^* can be defined in $\mathcal{E}(a, r)$ as the set χ_φ^* for some function $\varphi(z)$ holomorphic in this polydisk. By (4.1.17), in the neighborhood of each point $z_0 \in \chi_\varphi^*$ we have $\varphi(z) = [\psi_{z^0}(z)]^m$, where $\psi_{z^0}(z) \in \mathfrak{O}_{z^0}^{(n)}$, $m = \gamma_\varphi(z^0)$ and $\gamma_{\psi_{z^0}}(z^0) = 1$. A similar representation, with a function $\psi_{z^0}(z)$ such that $\psi_{z^0}(z^0) \neq 0$, holds also for $z^0 \notin \chi_\varphi$. In both cases, moreover, the function $\psi_{z^0}(z)$ is defined up to a factor $e^{2l\pi i/m}$, $l = 0, \cdots, n-1$. Since the set $\mathcal{E}(a, r) \setminus \{\chi_\varphi \setminus \chi_\varphi^*\}$ is simply connected,[12] it is readily seen that the functions $\psi_{z^0}(z)$ may be so chosen that they complete each other to a function $\psi(z)$ which is holomorphic throughout $\mathcal{E}(a, r)$, except perhaps at the singular points of the set χ_φ, and which is locally bounded in $\mathcal{E}(a, r)$. It is well known that the set of removable singularities is an analytic set of dimension lower than that of the space. Therefore the function $\psi(z)$ is holomorphically continuable to the entire polycylinder $\mathcal{E}(a, r)$. It is obvious that then $\psi^m(z) = \varphi(z)$, and so $\chi_\varphi = \chi_\psi$ and $\chi_\varphi^* = \chi_\psi^*$.

Let us now assume that for no fixed $z_1^0, \cdots, z_{n-1}^0$ does the function $\psi(z_1^0, \cdots, z_{n-1}^0, z_n)$ vanish identically, and let $\chi_\psi^{z_n}$ denote the set of points $z \in \chi_\psi$ at which $\partial\psi/\partial z_n = 0$. This set is analytic in the above-mentioned polydisk. We claim that

$$\dim \chi_\psi^{z_n} \leqq n - 2.$$

In view of the nature of the set χ_ψ, it will suffice to show that $\partial\psi/\partial z_n \not\equiv 0$ on χ_ψ^*. Suppose that this is not so, i.e. let $\partial\psi/\partial z_n = 0$ $\forall z \in \chi_\psi^*$. Then, since $\gamma_\psi(z) = 1$ $\forall z \in \chi_\psi^*$, at each point $z \in \chi_\psi^*$ there must be at least one nonzero derivative among $\partial\psi/\partial z_1, \cdots, \partial\psi/\partial z_{n-1}$. Let $z^0 \in \chi_\psi^*$, and suppose without loss of generality that $\partial\psi/\partial z_1|_{z=z^0} \neq 0$. Then it follows from the implicit function theorem that in some neighborhood of z^0 the set χ_ψ^* is given by

$$\chi_\psi^* = \{z: z_1 = \theta(z_2, \cdots, z_n)\},$$

where the function $\theta(z_2, \cdots, z_n)$ is holomorphic in a suitable neighborhood $\upsilon' \subset C^{n-1}$ of the point $(z_2^0, \cdots, z_{n-1}^0)$. Thus

$$\psi(\theta(z_2, \cdots, z_n), z_2, \cdots, z_n) = 0 \qquad \forall (z_2, \cdots, z_n) \in \upsilon'.$$

[11] See, for example, Hervé [1], or Gunning and Rossi [1].

[12] Because its complement has complex dimension $\leqq n - 2$.

Hence it follows that

$$\frac{\partial\psi}{\partial z_1}\frac{\partial\theta}{\partial z_n}+\frac{\partial\psi}{\partial z_n}=0 \qquad \forall(z_2,\cdots,z_n)\in\upsilon'.$$

Since $\partial\psi/\partial z_1\big|_{z=z^0}\neq 0$, while $\partial\psi/\partial z_n=0$ $\forall z\in\chi_\psi^*$, it follows that $\theta(z_2,\cdots,z_n)=\theta(z_2,\cdots,z_{n-1},z_n^0)$ $\forall(z_2,\cdots,z_n)\in\upsilon'$. In particular, $\theta(z_2^0,\cdots,z_{n-1}^0,z_n)\equiv\theta(z_2^0,\cdots,z_n^0)$, and so $\psi(z_1^0,\cdots,z_{n-1}^0,z_n)\equiv\psi(z^0)=0$. This contradicts our assumptions concerning $\psi(z)$ and completes the proof that $\partial\psi/\partial z_n\not\equiv 0$ on χ_ψ^*; hence $\dim\chi_\psi^{z^0}\leqq n-2$.

8. *Integration over the set* χ_f. Let $f(z)$ be a function analytic in the closed polydisk $\mathscr{E}_r=\mathscr{E}(0,r)$, and ω an exterior differential form of order $2n-2$ in $\mathscr{E}_r$. As we have shown, the analytic set $\chi_f\backslash\chi_f^*$ has dimension $n-2$, and hence the dimensions of the manifolds of which it consists cannot exceed $n-2$. Consequently the differential forms induced on these manifolds by the form ω are identically zero. Because of this, it is natural to set

$$\int\limits_{\chi_f\cap\mathscr{E}_r}\omega=\int\limits_{\chi_f^*\cap\mathscr{E}_r}\omega. \tag{4.1.20}$$

In turn, a natural definition of the integral $\int_{\chi_f^*\cap\mathscr{E}_r}\omega$ is

$$\int\limits_{\chi_f^*\cap\mathscr{E}_r}\omega=\sum_{j=1}^{N}\int^{S_j\cap\mathscr{E}_r}\omega,$$

where $S_1,\cdots,S_n$ are the components of χ_f^*. We mentioned previously that each component of χ_f^* is a manifold of complex dimension $n-1$, and hence the integrals $\int_{S_j}\omega$ are meaningful. Moreover, $N<\infty$ because the function $f(z)$ is holomorphic in the closure of our polydisk. Thus the final definition of the integral of the form ω over the set $\chi_f\cap\mathscr{E}_r$ is

$$\int\limits_{\chi_f\cap\mathscr{E}_r}\omega=\sum_{j=1}^{N}\int\limits_{S_j\cap\mathscr{E}_r}\omega. \tag{4.1.21}$$

Similarly we define the integral over the intersection of χ_f with $\mathscr{E}_r^*=\{z:|z_1|=r_1,\cdots,|z_{n-1}|=r_{n-1},|z_n|<r_n\}$ as follows. Let ω be a differential form of order $n-1$ on the manifold $\mathscr{E}_r^*$, and set $S_j^*=\mathscr{E}_r^*\cap S_j$, where S_j is the same as in (4.1.21). We now set

$$\int\limits_{\chi_f\cap\mathscr{E}_r^*}\omega=\sum_{j=1}^{N}\int\limits_{S_j^*}\omega.$$

In subsequent computations of integrals over the sets $\chi_f\cap\mathscr{E}_r$ and $\chi_f\cap\mathscr{E}_r^*$, we shall use the following lemmas.

LEMMA 4.1.1. *Let the function $g(z)$ be continuous in the polydisk $\mathscr{E}_r$, and let the form ω be defined in $\mathscr{E}_r$ by*

$$\omega = \alpha(z_1, \cdots, z_{n-1})\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1},$$

where the function $\alpha(z_1, \cdots, z_{n-1})$ *is continuous for* $|z_i| < r_i$, $i = 1, \cdots, n-1$. *Let* $z_n^{(j)}(z_1^0, \cdots, z_n^0)$ *be the points of intersection, arranged in order of increasing absolute values, of a plane* $\{z: z_1 = z_1^0, \cdots, z_{n-1} = z_{n-1}^0\}$, *in which* $f(z_1^0, \cdots, z_{n-1}^0, z_n) \not\equiv 0$, *with the set* $\chi_f \cap \mathscr{E}_r$, *and let* $\gamma_j^{(n)}(z_n^{(j)}; z_1^0, \cdots, z_{n-1}^0)$ *be the multiplicity of the root of the function* $f(z_1^0, \cdots, z_{n-1}^0, z_n)$ *at the point* $z_n^{(j)}$. *Then*

$$\int_{\mathscr{E}'_r} \sum_j g(z_1, \cdots, z_{n-1}, z_n^{(j)})\, \gamma_f^{(n)}(z_n^{(j)}; z_1, \cdots, z_{n-1})\, \omega = \int_{\chi_f \cap \mathscr{E}_r} \gamma_f(z) g(z) \omega, \tag{4.1.22}$$

where $\mathscr{E}'_r = \{(z_1, \cdots, z_{n-1}): |z_1| < r_1, \cdots, |z_{n-1}| < r_{n-1}\}$.

PROOF. By the definition of the integral $\int_{\chi_f}$ it is obviously sufficient to prove the lemma when the set χ_f^* is connected. In this case it follows from § 1.7 that $\gamma_f^{(n)}$ coincides with γ_f everywhere on χ_f, with the possible exception of some analytic set $\chi_f^{z_n}$ of dimension $\leqq n-2$. Moreover, we have $\chi_f^{z_n} \supset \chi_f \backslash \chi_f^*$. Let $\chi'_{f,n}$ denote the projection of $\chi_f^{z_n} \cap \mathscr{E}_r$ on the polydisk $\mathscr{E}'_r$. It is clear that the Lebesgue measure of the set $\chi'_{f,n}$ in C^{n-1} is zero.

Now let $\chi''_{f,n}$ denote the set of all points $(z'_1, \cdots, z'_{n-1}) \in \mathscr{E}'_r$ such that

$$\chi_f \cap \{z: z_1 = z'_1, \cdots, z_{n-1} = z'_{n-1}, |z_n| = r_n\} \neq \varnothing.$$

Suppose that for some fixed z_n^0 with $|z_n^0| = r_n$ we have $f(z_1, \cdots, z_{n-1}, z_n^0) \not\equiv 0$; then, as is readily seen, the Lebesgue measure of the set $\chi''_{f,n}$ is zero. Note that $f(z_1, \cdots, z_{n-1}, z_n^0) \equiv 0$ can hold only for finitely many points z_n^0 of the disk $|z_n| \leqq r_n$. Therefore without loss of generality we may assume that the Lebesgue measure of the set $\chi''_{f,n}$ is zero.

Set $\chi'_{f,n} = \chi'_{f,n} \cup \chi''_{f,n}$. From the above properties of the sets $\chi'_{f,n}$ and $\chi''_{f,n}$, it follows that

$$\int_{\mathscr{E}'_r} \sum_{|z_n^{(j)}| < r_n} g \gamma_f^{(n)} \omega = \int_{\mathscr{E}'_r \backslash \chi'_{f,n}} \sum_{|z_n^{(\rho)}| < r_n} g \gamma_f^{(n)} \omega \tag{4.1.23}$$

and

$$\int_{\chi_f \cap \mathscr{E}_r} g(z)\, \gamma_f \omega = \int_{\chi_f \backslash A} g \gamma_f^{(n)} \omega,$$

where

$$A = \{z: (z_1, \cdots, z_{n-1}) \in \mathscr{E}'_r \backslash \chi'_{f,n}, |z_n| < r_n\}. \tag{4.1.24}$$

Now consider any point $(z_1^*, \cdots, z_{n-1}^*) \in \mathscr{E}'_r \backslash \chi'_{f,n}$. The intersection

$$\chi_f \cap \{z: z_1 = z_1^*, \cdots, z_{n-1} = z_{n-1}^*, |z_n| < r_n\}$$

consists of finitely many points $(z_1^*, \cdots, z_{n-1}^*, z_n^{*(j)})$, $j = 1, \cdots, N$. Each such point

is in a component of the set $\chi_f \setminus \chi_f^{z_n}$. Take a neighborhood $\upsilon' \subset C^{n-1}$ of the point $(z_1^*, \cdots, z_{n-1}^*)$ so small that the parts of these components lying over υ' are disjoint, each such part being described by an equation $z_n = \varphi_j(z_1, \cdots, z_{n-1})$, where the function $\varphi_j(z_1, \cdots, z_{n-1})$ is holomorphic in υ' and $\varphi_j(z_1^*, \cdots, z_{n-1}^*) = z_n^{*(j)}$. That there exists such a neighborhood follows from the properties of the sets χ_f and $\chi_f \setminus \chi_f^{z_n}$ considered in § 1.7. Since $(z_1^*, \cdots, z_{n-1}^*) \notin \chi_{f,n}''$, the neighborhood υ' may also be chosen in such a way that

$$\chi_f \cap \{z\colon (z_1, \cdots, z_{n-1}) \in \upsilon', |z_n| < r_n\} = \bigcup_{j=1}^{N} \{z\colon z_n = \varphi_j(z_1, \cdots, z_{n-1})\}.$$

Then, since $\int_{\upsilon'} g(z_n^{(j)}; z_1, \cdots, z_{n-1})\,\omega$ is obviously the same as $\int g(z)\omega$ over the set $\{z\colon (z_1, \cdots, z_{n-1}) \in \upsilon',\ z_n = \varphi_j(z_1, \cdots, z_{n-1})\}$, we have

$$\int_{\upsilon'} \sum_{|z_j^{(n)}| < r_n} \gamma_f^{(n)}(z_n^{(j)}; z_1, \cdots, z_{n-1}) g(z_1, \cdots, z_{n-1}, z_n^{(j)})\,\omega = \int_{\upsilon'_{r_n} \cap \chi_f} \gamma_f^{(n)} g\omega,$$

where $\upsilon'_{r_n} = \upsilon' \times \{z_n\colon |z_n| < r_n\}$. Since the integral is a σ-additive set function, it follows from this equality that

$$\int_{\mathscr{E}'_r \setminus \chi'''_{f,n}} \sum_{|z_n^{(j)}| < r_n} \gamma_f^{(n)}(z_n^{(j)}; z_1, \cdots, z_{n-1}) g(z_1, \cdots, z_{n-1} z_n^{(j)})\omega = \int_{\chi_f \setminus A} \gamma_f^{(n)} g\omega, \tag{4.1.25}$$

where, as in (4.1.24), $A = \{z\colon (z_1, \cdots, z_{n-1}) \in \mathscr{E}'_r \setminus \chi'''_{f,n}, |z_n| < r_n\}$. The required (4.1.22) equality now follows from (4.1.23), (4.1.24) and (4.1.25). Q.E.D.

LEMMA 4.1.2. *Let $f(z)$, $\mathscr{E}_r$, $g(z)$, $z_n^{(j)}$ and $\gamma_f^{(n)}$ be the same as in Lemma* 4.1.1. *Let*

$$\mathscr{E}_r^* = \{z\colon |z_1| = r_1, \cdots, |z_{n-1}| = r_{n-1}, |z_n| < r_n\},$$
$$\mathscr{E}_r^{*\prime} = \{(z_1, \cdots, z_{n-1})\colon |z_i| = r_i, i = 1, \cdots, n-1\},$$

and let ω be an exterior form of order $n-1$ on $\mathscr{E}_r^*$ which is identical for any z_n, $|z_n| < r_n$, with the same form on $\mathscr{E}_r^{*\prime}$. Then

$$\int_{\mathscr{E}_r^{*\prime}} \sum_{|z_n^{(j)}| < r_n} g(z_1, \cdots, z_{n-1}, z_n^{(j)}) \gamma_f^{(n)}(z_n^{(j)}; z_1, \cdots, z_{n-1})\omega = \int_{\chi_f \cap \mathscr{E}_r^*} \gamma_f g\omega.$$

The proof is entirely similar to that of Lemma 4.1.1. and is therefore omitted.

In conclusion, we note that the above properties of analytic sets and integrals over analytic sets are obviously valid also for analytic sets in analytic manifolds. These sets are defined in a natural manner via local coordinates.

§ 2. Characteristics of the distribution of zeros of entire functions

In the one-variable case, the "density" of the zeros of an entire function $f(w)$ is most frequently characterized in terms of the functions $n_f(t)$ and $N_f(t)$, where $n_f(t)$

is the number of zeros (counting multiplicities) of $f(w)$ in the disk $|w| \leqq t$, and

$$N_f(t) = \int_0^\infty \frac{n_f(s) - n_f(0)}{s}\, ds + n_f(0) \ln t.$$

In this section we shall study the analogs of these functions in the several-variable case. From § 2.2 on, it will always be assumed that $f(0) = 0$, no special mention being made of this fact.

1. *The function* $n_f(t)$. Let $f(z)$ be an entire function in C^n. Let $f(w; \kappa)$, $w \in C^1$, denote the "restriction" of $f(z)$ to the ray $\kappa \in P_{n-1}$. In other words, if κ is defined by $\lambda \in C^n$, i.e.

$$\kappa = \{z: z_i = \lambda_i \omega,\ i = 1, \cdots, n\},$$

we set

$$f(w; \kappa) = f(w\lambda/|\lambda|). \tag{4.2.1}$$

We denote by $n_f(t; \kappa)$ the number of zeros (counting multiplicities) of the function $f(w; \kappa)$ in the disk $|w| \leqq t$. In view of (4.2.1) the function $n_f(t; \kappa)$ may also be defined as the number of zeros of $f(w\lambda)$ in the disk $\{w: |w| \leqq t/|\lambda|\}$. Note that if $f(0) \neq 0$ the function $n_f(t; \kappa)$ is bounded on P_{n-1} for any fixed t. Indeed, if $\kappa_j \to \kappa_0$ as $j \to \infty$, then in every disk $|w| \leqq t$ the sequence $f(w; \kappa_j)$ converges uniformly to the function $f(w; \kappa_0) \not\equiv 0$, and therefore the equality $\lim_{j\to\infty} n_f(t; \kappa_j) = \infty$ would contradict Hurwitz's theorem on the number of roots of the limit function.

The rate of growth of the function $n_f(t; \kappa)$ obviously characterizes the density of the zeros of $f(z)$ on the ray κ. In order to achieve such a characterization throughout the space C^n, it is natural to average the function $n_f(t; \kappa)$ over all rays κ, i.e. over the space P_{n-1}. Set[13]

$$n_f(t) = \frac{1}{W_{2n-2}} \int_{P_{n-1}} n_f(t; \kappa)\, dW_{2n-2},$$

where

$$W_{2n-2} = \int_{P_{n-1}} dW_{2n-2}.$$

It follows from Lemma 4.1.1 that

$$\int_{P_{n-1}} n_f(t; \kappa)\, dW_{2n-2} = \int_{\chi_f^t} \gamma_f(z) dW_{2n-2},$$

where $\chi_f = \{z: f(z) = 0\}$, $\chi_f^t = \chi_f \cap \{z: |z| \leqq t\}$, and $\gamma_f(z')$ is the multiplicity of the root of $f(z)$ at z'. Therefore

$$n_f(t) = \frac{1}{W_{2n-2}} \int_{\chi_f^t} \gamma_f(z) dW_{2n-2},$$

[13] The function $n_f(t)$ was first considered by Kneser [1]; see also Stoll [1], and Ronkin [12],[13].

and so the function $n_f(t)$ is essentially the mean projective area of the zero set of $f(z)$ in the ball $\{z: |z| \leqq t\}$.

THEOREM 4.2.1. *For any entire function $f(z)$ in C^n $(n > 1)$, the function $n_f(t)$ is continuous.*

PROOF. The theorem will clearly be proved if we show that for any $t > 0$

$$\int_{P_{n-1}} l(t; \kappa) dW_{2n-2} = 0, \tag{4.2.2}$$

where $l(t; \kappa)$ is the number of roots of $f(w; \kappa)$ on the circle $|w| = t$. Now (4.2.2) holds if and only if the set $\mathscr{A}_t = \chi_f \cap \{z: |z| = t\}$ has $(2n-2)$-dimensional Lebesgue measure zero on the sphere $S_t(0) = \{z: |z| = t\}$.[14] Suppose that this is false. Then, as is readily seen, there is a regular point z^0 of the set χ_f on the sphere $S_t(0)$ such that in any neighborhood of z^0 on $S_t(0)$ the intersection $S_t(0) \cap \chi_f$ has $(2n-2)$-dimensional Lebesgue measure zero. Since z^0 is a regular point of χ_f, it has a neighborhood V_{z^0} in C^n in which the set χ_f admits the repesentation

$$\chi_f \cap V_{z^0} = \left\{z: \underset{i=1,\cdots,n}{z_i = \varphi_i(\zeta)},\ \zeta \in C^{n-1},\ |\zeta| < 1\right\},$$

where the functions $\varphi_i(\zeta)$ are analytic in the ball $|\zeta| < 1$ and are such that $z_i^0 = \varphi_i(0)$, $i = 1, \cdots, n$, and the rank of their Jacobian is $n-1$. It follows from our assumptions on the set $\mathscr{A}_t \cap V_{z^0}$ that in the ball $|\zeta| < 1$ the set $\mathscr{A}'$ of points ζ at which

$$\sum_{i=1}^{n} \varphi_i(\zeta)\bar{\varphi}_i(\zeta) = t^2 \tag{4.2.3}$$

has positive $(2n-2)$-dimensional Lebesgue measure. The set $\mathscr{A}''$ of points of accumulation of $\mathscr{A}'$ also has positive Lebesgue measure. At each point $\zeta \in \mathscr{A}''$ the derivatives of the functions $\varphi_i(\zeta)$ are obviously uniquely determined by the values of these functions only on the set $\mathscr{A}'$. Therefore at each point of $\mathscr{A}''$ we have

$$\sum_{i=1}^{n} \varphi_i \frac{\partial \bar{\varphi}_i}{\partial \bar{\zeta}_j} = 0. \tag{4.2.4}$$

The rank of the Jacobian of the functions $\varphi_i(\zeta)$ is $n-1$. Without loss of generality we may assume that in the ball $|\zeta| < 1$

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial \zeta_1} & \cdots & \frac{\partial \varphi_{n-1}}{\partial \zeta_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial \varphi_1}{\partial \zeta_{n-1}} & \cdots & \frac{\partial \varphi_{n-1}}{\partial \zeta_{n-1}} \end{vmatrix} \neq 0,$$

[14] The l-dimensional Lebesgue measure of a compact set $K \subset R^m$ is defined as follows. Fix $\varepsilon > 0$, and consider all possible countable coverings of K by balls of radii $\leqq \varepsilon$. Let $m_\varepsilon(K)$ be the greatest lower bound over all these coverings of the sums $\omega_l \sum_j r_j^l$, where r_j are the radii of the balls in the covering. As ε decreases, $m_\varepsilon(K)$ increases monotonically, and hence $\lim_{\varepsilon \to 0} m_\varepsilon(K)$ exists; this limit is defined to be the l-dimensional Lebesgue measure of K.

and consequently equations (4.2.4) are solvable for the functions $\varphi_1\varphi_n^{-1}, \cdots, \varphi_{n-1}\varphi_n^{-1}$. Set $\varphi_i/\varphi_n = \psi_i$, and solve the system (4.2.4). It will follows that $\psi_i(\zeta) = \bar{F}_i(\zeta)$ on the set $\mathscr{A}''$, where $F_i(\zeta)$ are certain functions analytic in the ball $|\zeta| < 1$. The equality $\psi_i(\zeta) = \bar{F}_i(\zeta)$, $\zeta \in \mathscr{A}''$, implies that $\operatorname{Re} \psi_i(\zeta) = \operatorname{Re} F_i(\zeta)$ on $\mathscr{A}''$. Again using the fact that the derivatives of these functions are determined by their values at points of $\mathscr{A}''$, and also that the plurisubharmonic functions $\operatorname{Re} \psi_i$ and $\operatorname{Re} F_i$ are analytic functions of the real variables $\operatorname{Re} \zeta_j$ and $\operatorname{Im} \zeta_j$, $j = 1, \cdots, n$, we conclude that $\operatorname{Re} \psi_i(\zeta) = \operatorname{Re} F_i(\zeta)$ everywhere in the ball $|\zeta| < 1$. Consequently $\operatorname{Im} \psi_i(\zeta) = \operatorname{Im} F_i(\zeta) + \text{const}$. However, on $\mathscr{A}''$ we have $\psi_i(\zeta) = \bar{F}_i(\zeta)$, and so in the ball $|\zeta| < 1$ we have $\operatorname{Im} \psi_i(\zeta) \equiv \text{const}$, whence it follows immediately that $\psi_i(\zeta) \equiv \text{const}$ in this ball. Thus there exist constants $c_1, \cdots, c_{n-1}$ such that

$$\varphi_i(\zeta) = c_i\varphi_n(\zeta) \qquad \forall \zeta \in \{\zeta\colon |\zeta| < 1\}, \qquad i = 1, \cdots, n-1.$$

Hence by (4.2.3) it follows that $|\varphi_n(\zeta)| \equiv \text{const}$, and so the function $\varphi_n(\zeta)$, like the functions $\varphi_1, \cdots, \varphi_{n-1}$, is also identically constant; this clearly contradicts our assumptions concerning these functions. Q.E.D.

We now establish a connection between the function $n_f(t)$ and the Riesz mass distribution of the function $\ln|f(z)|$. Following the notation of Chapter 1, § 2, we denote the measure of the ball $\{z\colon |z| \leqq t\}$ associated with the function $\ln|f(z)|$ by $\mu_f(t)$:

$$\mu_f(t) = \frac{1}{\theta_{2n}} \int\limits_{|z| \leqq t} \Delta \ln|f|\, dV_{2n}.$$

The function $\Delta \ln|f|$ is of course treated here as a distribution. We also set

$$\nu_f(t) = \frac{\theta_{2n}}{\sigma_{2n}} \frac{\mu_f(t)}{t^{2n-2}}.^{15)}$$

THEOREM 4.2.2. *For any entire function $f(z)$, $\nu_f(t) = n_f(t)$ $(t > 0)$.*

PROOF. We first prove a few lemmas.

LEMMA 4.2.1. *Let $\varphi(w)$ be an entire function in C^1. Then*

$$n_\varphi(t) = \frac{1}{2\pi} \int\limits_{|w| \leqq t} \Delta \ln|\varphi| dV_2.$$

PROOF. Any function $\varphi(w)$ analytic in the disk $|w| \leqq t$ admits in this disk the representation

$$\varphi(w) = \psi(w) \prod_{j=1}^{p} (w - w_j)^{\gamma_j},$$

where w_j are all the distinct roots of $\varphi(w)$ in the disk $|w| \leqq t$, γ_j is the multiplicity

15) The constants θ_{2n} and σ_{2n} are the same as in Chapter 1.

of w_j, and the function $\psi(w)$ is analytic in $|w| \leqq t$ and nonvanishing there. Hence by the equality

$$\frac{1}{2\pi} \Delta \ln|w| = \delta(w)$$

(see § 2, Chapter 1) it follows that

$$\frac{1}{2\pi} \int_{|w|\leqq t} \Delta \ln|\varphi| \, dV_2 = \frac{1}{2\pi} \int_{|w|\leqq t} \sum_{j=1}^{p} \gamma_j \Delta \ln|w - w_j| \, dV_2$$

$$= \sum_{j=1}^{p} \gamma_j \int_{|w|\leqq t} \delta(w - w_j) dV_2 = n_\varphi(t).$$

This proves the lemma.

LEMMA 4.2.2.[16] *For any twice continuously differentiable function $u(z)$ in C^n,*

$$\frac{4}{(n-1)!} \int_{|z|<t} \alpha \wedge \beta^{n-1} = \frac{1}{t^{2n-2}} \int_{|z|<t} \Delta u dV_{2n},$$

where

$$\alpha = \frac{i}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

and

$$\beta = \frac{i}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \ln(z, \bar{z})}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

PROOF. A direct check shows that

$$\int_{|z|<t} \Delta u dV_{2n} = 4\left(\frac{i}{2}\right)^n \int_{|z|<t} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

$$= \frac{4}{(n-1)!} \int_{|z|<t} \alpha \wedge \sigma^{n-1}, \tag{4.2.5}$$

where

$$\sigma = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

It is also immediate that $\alpha = d\theta$, where

$$\theta = \frac{i}{4} \sum_{j=1}^{n} \left(\frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j - \frac{\partial u}{\partial z_j} dz_j \right).$$

Hence $d\alpha = 0$, $d\beta = 0$, and

$$d(\theta \wedge \sigma^{n-1}) = d\theta \wedge \sigma^{n-1} = \alpha \wedge \sigma^{n-1}. \tag{4.2.6}$$

By Stokes' formula it now follows that

$$\int_{|z|<t} \alpha \wedge \sigma^{n-1} = \int_{|z|=t} \theta \wedge \sigma^{n-1}. \tag{4.2.7}$$

In view of (4.2.5) and (4.2.7) we now see that

[16] This is a special case of a proposition of Lelong [5].

$$\int_{|z|<t} \Delta u dV_{2n} = \frac{4}{(n-1)!} \int_{|z|=t} \theta \wedge \sigma^{n-1}. \tag{4.2.8}$$

On the other hand, since $d\beta = 0$,

$$\alpha \wedge \beta^{n-1} = d\theta \wedge \beta^{n-1} = d(\theta \wedge \beta^{n-1}),$$

and so

$$\int_{|z|<t} \alpha \wedge \beta^{n-1} = \int_{|z|=t} \theta \wedge \beta^{n-1}. \tag{4.2.9}$$

Elementary manipulations now show that

$$\beta = -\frac{i}{2} \frac{\sum_1^n \bar{z}_p dz_p \wedge \sum_1^n z_q d\bar{z}_q}{|z|^4} + \frac{\sigma}{|z|^2}. \tag{4.2.10}$$

It is readily seen that if $\langle z, \bar{z} \rangle \equiv \text{const}$, then

$$\sum_{p=1}^n \bar{z}_p dz_p \wedge \sum_{q=1}^n z_q d\bar{z}_q = 0.$$

On the sphere $\{z: |z| = t\}$, therefore, we have

$$\beta^{n-1} = \sigma^{n-1}/t^{2n-2}. \tag{4.2.11}$$

From (4.2.11) and (4.2.9) it follows that

$$\int_{|z|<t} \alpha \wedge \beta^{n-1} = \frac{1}{t^{2n-2}} \int_{|z|=t} \theta \wedge \sigma^{n-1}. \tag{4.2.12}$$

Further, comparing (4.2.12) and (4.2.8), we see that

$$\frac{1}{t^{2n-2}} \int_{|z|<t} \Delta u dV_{2n} = \frac{4}{(n-1)!} \int_{|z|<t} \alpha \wedge \beta^{n-1}.$$

This completes the proof.

Now consider the function $\ln|f(z)|$, where $f(z)$ is the function figuring in the statement of our theorem. This function is plurisubharmonic in C^n, and therefore

$$\ln|f(z)| = \lim_{m\to\infty} u_m(z),$$

where $u_m(z)$ is a monotone decreasing sequence of infinitely differentiable plurisubharmonic functions. By the properties of distributions and integrals of distributions presented in the Introduction, it follows that for all $\lambda \in C^n$ and $t > 0$ such that $f(\lambda w) \not\equiv 0$ for $|w| = t/|\lambda|$ we have

$$\int_{|w|<t/|\lambda|} \Delta_w \ln|f(\lambda w)| dV_2 = \lim_{m\to\infty} \int_{|w|<t/|\lambda|} \Delta_w u_m(\lambda w) dV_2. \tag{4.2.13}$$

Earlier, while proving the continuity of $n_f(t)$, we showed that the set of all $\kappa \in P_{n-1}$ for which $n_f(t; \kappa) - n_f(t-0; \kappa) > 0$ has Lebesgue measure zero in P_{n-1}. Therefore when $\lambda_n = 1$ equality (4.2.13) can fail to hold only on a set of Lebesgue measure zero in $C_{(\lambda)}^{n-1}$. Moreover, as is readily seen, for any fixed t the integrals on

the right of this equality, divided by $|\lambda|^{2n}$, have an integrable majorant in $C^{n-1}_{(\lambda)}$.[17] Using this remark, Lemma 4.2.1 and (4.2.13) and defining a ray κ by the points λ such that $\lambda_n = 1$, we get

$$
\begin{aligned}
n_f(t) &= \frac{1}{W_{2n-2}} \int_{P_{n-1}} n_f(t;\kappa)\, dW_{2n-2} \\
&= \frac{1}{2\pi W_{2n-2}} \int_{P_{n-1}} \left(\int_{|w|<t/|\lambda|} \Delta_w \ln|f(\lambda w)| dV_2 \right) dW_{2n-2} \\
&= \frac{2(i/2)^n}{\pi W_{2n-2}} \lim_{m\to\infty} \int_{C^{n-1}_{(\lambda)}} \int_{|w|<t/|\lambda|} \frac{\partial^2 u_m(\lambda w)}{\partial w \partial \bar{w}} \frac{dw \wedge d\bar{w} \wedge d\lambda_1 \wedge d\bar{\lambda}_1 \wedge \cdots \wedge d\bar{\lambda}_{n-1}}{(1 + \sum_{p=1}^{n-1} |\lambda_p|^2)^n} \\
&= \frac{2(i/2)^n}{\pi W_{2n-2}} \lim_{m\to\infty} \int_{C^{n-1}_{(\lambda)}} \int_{|w|<t/|\lambda|} \frac{\sum_{i,j=1}^{n} \frac{\partial^2 u_m}{\partial z_i \partial \bar{z}_j} \lambda_i \bar{\lambda}_j}{(1 + \sum_{p=1}^{n-1} |\lambda_p|^2)^n} dw \wedge d\bar{w} \wedge d\lambda_1 \\
&\qquad \wedge d\bar{\lambda}_1 \wedge \cdots \wedge d\bar{\lambda}_{n-1}.
\end{aligned}
$$

Perform the substitution $z_1 = \lambda_1 w, \cdots, z_{n-1} = \lambda_{n-1} w, z_n = w$ in the last integral. After a few simple manipulations, we get

$$
n_f(t) = \left(\frac{i}{2}\right)^n \frac{2}{\pi W_{2n-2}} \lim_{m\to\infty} \int_{|z|<t} \frac{\sum_{i,j=1}^{n} \frac{\partial^2 u_m}{\partial z_i \partial \bar{z}_j} z_i \bar{z}_j}{|z|^{2n}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n. \tag{4.2.14}
$$

This integral can now be expressed as

$$
\left(\frac{2}{i}\right)^n \frac{1}{(n-1)!} \int_{|z|<t} \alpha \wedge \beta^{n-1},
$$

where the differential forms α and β are as in Lemma 4.2.2. Indeed,

$$
\begin{aligned}
\left(\frac{2}{i}\right)^n \frac{1}{(n-1)!} \int_{|z|<t} \alpha \wedge \beta^{n-1} &= \frac{1}{(n-1)!} \int_{|z|<t} \frac{2}{i} \alpha \wedge \left\{ \sum_{i,j=1}^{n} \frac{\partial^2 \ln\langle z, \bar{z}\rangle}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \right\}^{n-1} \\
&= \frac{1}{(n-1)!} \int_{|z|<t} \frac{2}{i} \alpha \wedge \left\{ \frac{1}{|z|^2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j - \frac{1}{|z|^4} \sum_{i,j=1} \bar{z}_i z_j dz_i \wedge d\bar{z}_j \right\}^{n-1} \\
&= \frac{1}{(n-1)!} \int_{|z|<t} \left\{ \sum_{i,j=1}^{n} \frac{\partial^2 u_m}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \right\} \qquad (4.2.15) \\
&\wedge \left\{ \frac{1}{|z|^{2n-2}} \left(\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \right)^{n-1} - \frac{n-1}{|z|^{2n}} \sum_{i=1}^{n} \bar{z}_i dz_i \wedge \sum_{j=1}^{n} z_j d\bar{z}_j \wedge \left(\sum_{p=1}^{n} dz_p \wedge d\bar{z}_p \right)^{n-2} \right\} \\
&= \int_{|z|<t} \frac{1}{|z|^{2n-2}} \sum_{i,j=1}^{n} \frac{\partial^2 u_m}{\partial z_i \partial \bar{z}_j} z_i \bar{z}_j dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\end{aligned}
$$

[17] Let $\varphi(w)$ be an infinitely differentiable function equal to 1 for $|w|<t$, having values between 0 and 1 for $t \leqq |w| \leqq 2t$, and vanishing for $|w| > 2t$. Then

$$
\begin{aligned}
\int_{|w|<t/|\lambda|} \Delta_w u_m(\lambda w) dV_2 &\leqq \int_{|w|<\infty} \varphi(|\lambda| w) \Delta_w u_m(\lambda w) dV_2 \\
&= \int_{|w|<\infty} u_m(\lambda w) \Delta_w \varphi(|\lambda| w) dV_2 \geqq \text{const} \cdot |\lambda|^2 \int_{|w|<2t/|\lambda|} |u_m(\lambda w)| dV_2.
\end{aligned}
$$

The required majorant is the following function of λ ($\lambda_n = 1$):

$$
\frac{\text{const}}{|\lambda|^{2n-2}} \int_{|w|<2t/|\lambda|} (|u_1(\lambda w)| + |\ln|f(\lambda w)||) dV_2.
$$

Now, by Lemma 4.2.2,

$$\frac{4}{(n-1)!}\int_{|z|<t}\left\{\sum_{i,j=1}^{n}\frac{\partial^2 u_m}{\partial z_i\partial\bar{z}_j}dz_i\wedge d\bar{z}_j\right\}\wedge\left\{\sum_{i,j=1}^{n}\frac{\partial^2\ln\langle z,\bar{z}\rangle}{\partial z_i\partial\bar{z}_j}dz_i\wedge d\bar{z}_j\right\}^{n-1}$$
$$=\frac{1}{t^{n-2}}\int_{|z|<t}\Delta u_m dz_1\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n. \tag{4.2.16}$$

It follows immediately from (4.2.16), (4.2.15) and (4.2.14) that

$$n_f(t)=\frac{1}{2\pi W_{2n-2}}\lim_{m\to\infty}\frac{1}{t^{2n-2}}\int_{|z|<t}\Delta u_m(z)dV_{2n}.$$

But

$$\theta_{2n}\mu_f(t-0)\leqq\lim_{m\to\infty}\int_{|z|<t}\Delta u_m(z)dV_{2n}\leqq\int_{|z|\leqq t}\Delta\ln|f|dV_{2n}=\theta_{2n}\mu_f(t),$$

and since the functions $n_f(t)$ and $\mu_f(t)$ are monotone, and $n_f(t)$ is even continuous, we get

$$n_f(t)=\frac{\theta_{2n}}{2\pi W_{2n-2}}\frac{\mu_f(t)}{t^{2n-2}}=\frac{\sigma_{2n}}{2\pi W_{2n-2}}\nu_f(t)=\nu_f(t).$$

Q.E.D.

REMARK. Theorem 4.2.2 is obviously also valid when the function $f(z)$ is not necessarily entire but analytic in a ball $\{z:|z|<r\}$ with $r>t$.

We have already mentioned the simple geometric interpretation of $n_f(t)$: $W_{2n-2}\cdot n_f(t)$ is the projective area of the intersection of the ball $|z|<t$ and the analytic set $\chi_f=\{z:f(z)=0\}$. An even simpler interpretation can be given for $\mu_f(t)$. We claim that

$$\mu_f(t)=\frac{2\pi}{\theta_{2n}}\int_{\chi_f^t}\gamma_f(z)dV_{2n-2} \tag{4.2.17}$$

(see Lelong [2]), where $\chi_f^t=\chi_f\cap\{z:|z|\leqq t\}$ and dV_{2n-2} is the volume element of $(2n-2)$-dimensional area of the manifold χ_f. In other words, we wish to show that $(\theta_{2n}/2\pi)\mu_f(t)$ *is the* $(2n-2)$*-dimensional area of that part of the set* χ_f *lying in the ball* $|z|\leqq t$, *calculated counting multiplicities.* To prove this, we observe that, by Lemma 4.2.1, for any polydisk $\mathscr{E}_r=\{z:|z_i|<r_i,\ i=1,\cdots,n\}$,

$$\left(\frac{i}{2}\right)^n\int_{\mathscr{E}_r}\Delta\ln|f|dz_1\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$$
$$=\left(\frac{i}{2}\right)^n 4\sum_{j=1}^{n}\int_{\substack{|z_p|\leqq r_p\\ p\neq j}}\prod_{p\neq j}dz_p\wedge d\bar{z}_p\int_{|z_j|\leqq r_j}\frac{\partial^2\ln|f|}{\partial z_j\partial\bar{z}_j}dz_j\wedge d\bar{z}_j \tag{4.2.18}$$
$$=2\pi\left(\frac{i}{2}\right)^{n-1}\sum_{j=1}^{n}\int_{\substack{|z_p|\leqq r_p\\ p\neq j}}n_j(r_j;z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_n)\prod_{p\neq j}dz_p\wedge d\bar{z}_p,$$

where n_j denotes the number of roots of $f(z)$ in the disk $|z_j| \leqq r_j$ for fixed values of $z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n$. By Lemma 4.1.1, moreover,

$$\int\limits_{\substack{|z_p| \leqq r_p \\ p \neq j}} n_j \prod_{p \neq j} dz_p \wedge d\bar{z}_p = \int\limits_{\chi_f \cap \bar{\mathscr{E}}_r} \gamma_f(z) \prod_{p \neq j} dz_p \wedge d\bar{z}_p.$$

By (4.2.18) this implies that

$$\int\limits_{\mathscr{E}_r} \Delta \ln|f| dV_{2n} = 2\pi \left(\frac{i}{2}\right)^{n-1} \int\limits_{\chi_f \cap \bar{\mathscr{E}}_r} \gamma_f(z) \sum_{j=1}^{n} \prod_{p \neq j} dz_p \wedge d\bar{z}_p$$

$$= \frac{2\pi}{(n-1)!} \int\limits_{\chi_f \cap \bar{\mathscr{E}}_r} \gamma_f(z) \left(\frac{i}{2} \sum_{p=1}^{n} dz_p \wedge d\bar{z}_p\right)^{n-1} = 2\pi \int\limits_{\chi_f \cap \bar{\mathscr{E}}_r} \gamma_f(z) dV_{2n-2}.$$

Since this relation is valid in an arbitrary polydisk $\mathscr{E}_r$, the analogous relation holds in a ball:

$$\frac{1}{2\pi} \int\limits_{|z| \leqq t} \Delta \ln|f| dV_{2n} = \int\limits_{\chi_f^t} \gamma_f(z) dV_{2n-2}.$$

Hence by the definition of the measure μ we immediately get (4.2.17). Thus the Riesz measure associated with $\ln|f(z)|$ is proportional to the $(2n-2)$-dimensional area of its zero surface.

Theorem 4.2.2 and the above remarks on the measure associated with the logarithm of the modulus of an analytic function imply the following important proposition (see Rutishauser [1] or Lelong [2], [5]):

If the function $f(z)$ is analytic in the ball $\{z: |z| \leqq t\}$ and $f(0) = 0$, then the $(2n-2)$-dimensional area $V_{2n-2}(t)$ of the set χ_f^t satisfies the following estimate:

$$V_{2n-2}(t) \geqq \omega_{2n-2} \gamma_f(0) t^{2n-2}.^{18)} \tag{4.2.19}$$

Indeed, it follows from the definition of $n_f(t)$ that $n_f(t) \geqq \gamma_f(0)$ for all $t \geqq 0$. On the other hand, by Theorem 4.2.2 and (4.2.17),

$$\frac{V_{2n-2}(t)}{t^{2n-2}} = \frac{\theta_{2n} \mu_f(t)}{2\pi t^{2n}} = \frac{\sigma_{2n}}{2\pi} \nu_f(t) = \omega_{2n-2} n_f(t),$$

and so this function is monotone nondecreasing; this implies (4.2.19).

2. *The function $N_f(t)$.* Instead of the functions $n_f(t)$ and $n_f(t; \kappa)$, it is sometimes more convenient to characterize the set χ_f in terms of their "averages" $N_f(t)$ and $N_f(t; \kappa)$, defined by

$$N_f(t) = \int_0^t \frac{n_f(s)}{s} ds \tag{4.2.20}$$

---

18) As before, ω_{2n-2} is the volume of the ball $\{x: x \in R^{2n-2}, |x| < 1\}$.

and

$$N_f(t;\kappa)=\int_0^t \frac{n_f(s;\kappa)}{s}\,ds.^{19)}$$

These functions (see the Introduction, § 4) have the same order and type (minimal, normal or maximal) as $n_f(t)$ and $n_f(t;\kappa)$, respectively, and also belong to the same convergence class. The reason for their convenience is that simple analytic expressions are available for them. Indeed, applying Jensen's formula to the function $f(w;\kappa)$, we get

$$\begin{aligned}N_f(t;\kappa)&=\frac{1}{2\pi}\int_0^{2\pi}\ln|f(te^{i\varphi};\kappa)|d\varphi-\ln|f(0)|\\&=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f\left(\frac{t\lambda_1e^{i\varphi}}{|\lambda|},\cdots,\frac{t\lambda_ne^{i\varphi}}{|\lambda|}\right)\right|d\varphi-\ln|f(0)|.\end{aligned}\tag{4.2.21}$$

To derive a similar expression for $N_f(t)$, we first note that, as is easily seen,

$$N_f(t)=\frac{1}{W_{2n-2}}\int_{P_{n-1}}N_f(t;\kappa)dW_{2n-2}.$$

Setting $\lambda=(\lambda_1,\cdots,\lambda_{n-1},1)$ in (4.2.21) and using (4.1.16), we see that

$$N_f(t)=\frac{1}{2\pi W_{2n-2}}\int_{C^{n-1}}\frac{dV_{2n-2}}{|\lambda|^{2n}}\int_0^{2\pi}\ln\left|f\left(\frac{t\lambda e^{i\varphi}}{|\lambda|}\right)\right|d\varphi-\ln|f(0)|.\tag{4.2.22}$$

We shall use these formulas to estimate the function $N_f(t;\kappa)$.

THEOREM 4.2.3(RONKIN [12]). *For any $\delta>1$ there exists a constant C_δ such that for all $t>0$ and any $\kappa\in P_{n-1}$*

$$N_f(t;\kappa)\leqq C_\delta N_f(\delta t).\tag{4.2.23}$$

PROOF. We first prove (4.2.23) for the ray κ_0 defined by the point $\lambda_0=(0,\cdots,0,1)$. To abbreviate the notation, we set $(\lambda_1,\cdots,\lambda_{n-1})=\lambda^*$, $\lambda=(\lambda^*,1)$. Since the function $N_f(t;\kappa)$ is nonnegative and monotone nondecreasing, it follows from (4.2.22) that for any $\varepsilon>0$

$$\begin{aligned}N_f(t)&\geqq\frac{1}{2\pi W_{2n-2}}\int_{|\lambda^*|<\varepsilon}\left\{\int_0^{2\pi}\ln\left|f\left(\frac{\lambda te^{i\varphi}}{|\lambda|}\right)\right|d\varphi-2\pi\ln|f(0)|\right\}\frac{dV_{2n-2}}{|\lambda|^{2n}}\\&\geqq\frac{(1+\varepsilon^2)^{-n}}{2\pi W_{2n-2}}\int_{|\lambda^*|<\varepsilon}\left\{\int_0^{2\pi}\ln\left|f\left(\frac{t\lambda e^{i\varphi}}{\sqrt{1+\varepsilon^2}}\right)\right|d\varphi-2\pi\ln|f(0)|\right\}dV_{2n-2}\\&=\frac{(1+\varepsilon^2)^{-n}}{2\pi W_{2n-2}}\int_0^{2\pi}d\varphi\int_{|\lambda^*|<\varepsilon}\left\{\ln\left|f\left(\frac{t\lambda e^{i\varphi}}{\sqrt{1+\varepsilon^2}}\right)\right|-\ln|f(0)|\right\}dV_{2n-2}.\end{aligned}$$

19) Recall the assumption, adopted at the beginning of this section, that henceforth $f(0)\neq0$.

Now $\ln\left|f(t\lambda e^{i\varphi}/\sqrt{1+\varepsilon^2})\right|$ is a plurisubharmonic function of $\lambda_1, \cdots, \lambda_{n-1}$, and so

$$\frac{1}{\omega_{2n-2}\varepsilon^{2n-2}} \int_{|\lambda^*|<\varepsilon} \ln\left|f\left(\frac{e^{i\varphi}t\lambda}{\sqrt{1+\varepsilon^2}}\right)\right| dV_{2n-2} \geqq \ln\left|f(0, \cdots, 0, te^{i\varphi}/\sqrt{1+\varepsilon^2})\right|.$$

Therefore

$$N_f(t) \geqq \frac{\omega_{2n-2}}{2\pi W_{2n-2}} \cdot \frac{\varepsilon^{2n-2}}{(1+\varepsilon^2)^n} \left\{\int_0^{2\pi} \ln\left|f\left(0, \cdots, 0, \frac{te^{i\varphi}}{\sqrt{1+\varepsilon^2}}\right)\right| d\varphi - 2\pi \ln|f(0)|\right\}$$

$$= \frac{\omega_{2n-2}}{W_{2n-2}} \cdot \frac{\varepsilon^{2n-2}}{(1+\varepsilon^2)^n} N_f\left(\frac{t}{\sqrt{1+\varepsilon^2}}; \kappa_0\right), \tag{4.2.24}$$

and hence $N_f(t; \kappa_0) \leqq C_\delta N_f(\delta t)$, where

$$C_\delta = \frac{W_{2n-2}}{\omega_{2n-2}} \frac{\delta^{2n}}{(\delta^2-1)^{n-1}}.$$

We have thus proved (4.2.23) for $\kappa = \kappa_0$. To prove the estimate for arbitrary κ, we observe that the functions $n_f(t)$ and $N_f(t)$ are clearly invariant under analytic "rotations," i.e. unitary transformations of the space C^n. Let $\zeta = \alpha(z)$ be an analytic "rotation" taking the ray κ to κ_0. Consider the function $\varphi(z) = f(\alpha(z))$. By the foregoing arguments,

$$N_\varphi(t; \kappa_0) \leqq C_\delta N_f(\delta t). \tag{4.2.25}$$

On the other hand, it follows from the definition of the transformation $\alpha(z)$ that $\varphi(w; \kappa_0) = f(w; \kappa)$ and hence

$$N_\varphi(w; \kappa_0) = N_f(w; \kappa). \tag{4.2.26}$$

In addition, by the invariance of the function $N_f(t)$ under analytic "rotations,"

$$N_f(t) = N_\varphi(t). \tag{4.2.27}$$

It now follows from (4.2.25), (4.2.26) and (4.2.27) that $N_f(t; \kappa) \leqq C_\delta N_f(\delta t)$ for any κ and all $t > 0$. Q.E.D.

COROLLARY. *For any $\delta > 1$ there exists a constant C_δ such that, for all $\kappa \in P_{n-1}$ and all t greater than some $t = t_0$,*

$$n_f(t; \kappa) \leqq C_\delta n_f(t\delta) \ln(t/t_0).$$

3. *The functions $n_{f,\tau}(t)$ and $N_{f,\tau}(t)$.* The functions to be considered here characterize the distribution of the points at which the set χ_f intersects a certain special cone.

Let $\tau = (\tau_1, \cdots, \tau_n)$, where $\tau_i > 0$, $i = 1, \cdots, n$. Let P_τ denote the manifold formed in P_{n-1} by all complex rays

$$\kappa = \left\{z: \underset{j=1,\cdots,n-1}{z_j = \tau_j e^{i\varphi_j} w},\ z_n = \tau_n w, |w| < \infty\right\}$$

where $\varphi_j \in [0, 2\pi]$, $j = 1, \cdots, n-1$. Note that the set of points z in C^n lying in the complex rays $\kappa \in P_\tau$ is defined by the condition

$$|z_1|/\tau_1 = \cdots = |z_n|/\tau_n.$$

Let dW_τ denote the volume element of the $(n-1)$-dimensional manifold P_τ in the metric induced by that of P_{n-1}. We claim that

$$dW_\tau = \frac{\tau_1 \cdots \tau_n}{|\tau|^n} d_{n-1}\varphi, \tag{4.2.28}$$

where $d_{n-1}\varphi$ is the volume element in the euclidean space of the variables $\varphi_1, \cdots, \varphi_{n-1}$. Indeed, as was shown in § 1, the line element in P_{n-1} is defined by

$$d\sigma^2 = \frac{(\sum_1^n |z_j|^2)(\sum_1^n dz_j d\bar{z}_j) - (\sum_1^n z_j d\bar{z}_j)(\sum_1^n \bar{z}_j dz_j)}{|z|^4}.$$

Substituting in this formula $z_j = \tau_j e^{i\varphi_j}$ for $j = 1, \cdots, n-1$ and $z_n = \tau_n$, we find the line element $d\sigma_\tau$ on the manifold P_τ:

$$d\sigma_\tau^2 = \frac{|\tau|^2 \sum_1^{n-1} \tau_j^2 d\varphi_j^2 - \sum_{i,j=1}^{n-1} \tau_i^2 \tau_j^2 d\varphi_i d\varphi_j}{|\tau|^4} = \sum_{i,j=1}^{n-1} G_{i,j} d\varphi_i d\varphi_j,$$

where

$$G_{i,j} = -\tau_i^2 \tau_j^2 / |\tau|^4 \qquad \text{for } i \neq j$$

and

$$G_{i,i} = \frac{\tau_i^2 \sum_{j \neq i} \tau_j^2}{|\tau|^4}, \qquad i = 1, \cdots, n-1.$$

By § 1, the volume element dW_τ on P_τ is given by the formula

$$dW_\tau = \sqrt{\det\{G_{i,j}\}}\, d_{n-1}\varphi.$$

An elementary computation shows that in this case

$$\det\{G_{i,j}\} = \tau_1^2 \cdots \tau_n^2 / |\tau|^{2n},$$

and this proves (4.2.28).

It follows immediately from (4.2.28) that

$$W_\tau = \frac{\tau_1 \cdots \tau_n}{|\tau|^n} (2\pi)^{n-1}, \tag{4.2.29}$$

where W_τ is the volume of the entire manifold P_τ.

Now set

$$n_{f,\tau}(t) = \frac{1}{W_\tau} \int_{P_\tau} n_f(t; \kappa) dW_\tau, \qquad N_{f,\tau}(t) = \frac{1}{W_\tau} \int_{P_\tau} N_f(t; \kappa) dW_\tau.$$

It is readily seen that the last equality is equivalent to

$$N_{f,\tau}(t) = \int_0^t \frac{n_{f,\tau}(s)}{s}\, ds.$$

The functions $n_{f,\tau}(t)$ and $N_{f,\tau}(t)$ characterize the distribution of those points $z \in \chi_f$ for which $|z_1|/\tau_1 = \cdots = |z_n|/\tau_n$. We denote the set of all such points by $\chi_{f,\tau}$. It can be shown that in a certain sense the points of $\chi_{f,\tau}$ are distributed as densely as those of the set χ_f. This statement is made precise in the following assertion.

THEOREM 4.2.4 (RONKIN [12]). *For any τ, the functions $n_{f,\tau}(t)$, $N_{f,\tau}(t)$, $n_f(t)$ and $N_f(t)$ have the same order and type (minimal, normal or maximal) and belong to the same convergence class.*

PROOF. It follows from § 4 of the Introduction that we need only prove the assertion for $N_{f,\tau}(t)$ and $N_f(t)$. We first estimate $N_{f,\tau}(t)$ from above. Applying Theorem 4.2.3, we get

$$N_{f,\tau}(t) = \frac{1}{W_\tau} \int_{P_\tau} N_f(t;\kappa)\, dW_\tau \leqq \frac{1}{W_\tau} \int_{P_\tau} C_\delta N_f(\delta t)\, dW_\tau = C_\delta N_f(\delta t). \tag{4.2.30}$$

We now estimate $N_{f,\tau}(t)$ from below. It follows from (4.2.28), (4.2.29) and (4.2.21) that

$$N_{f,\tau}(t) = \left(\frac{1}{2\pi}\right)^{n-1} \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(\frac{t\tau_1 e^{i(\varphi_1+\varphi)}}{|\tau|}, \ldots, \frac{t\tau_n e^{i(\varphi_{n-1}+\varphi)}}{|\tau|}, \frac{t\tau_n e^{i\varphi}}{|\tau|} \right) \right| d\varphi \right\} d_{n-1}\varphi - \ln|f(0)|.$$

Perform the substitution $\varphi_i + \varphi = \psi_i$, $i = 1, \cdots, n-1$, $\varphi = \psi_n$ in the integral figuring in this formula. The result is

$$N_{f,\tau}(t) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \ln \left| f\left(\frac{t\tau_1 e^{i\psi_1}}{|\tau|}, \ldots, \frac{t\tau_n e^{i\psi_n}}{|\tau|} \right) \right| d_n\psi - \ln|f(0)|, \tag{4.2.31}$$

where $d_n\psi = d\psi_1 \cdots d\psi_n$.

On the other hand, introducing polar coordinates in each coordinate plane of C^{n-1}, we get[20]

$$\begin{aligned} N_f(t) &= \frac{1}{2\pi W_{2n-2}} \int_{C^{n-1}} \frac{dV_{2n-2}}{|\lambda|^{2n}} \int_0^{2\pi} \ln \left| f\left(\frac{te^{i\varphi}\lambda}{|\lambda|} \right) \right| d\varphi - \ln|f(0)| \\ &= \frac{(2\pi)^{n-1}}{W_{2n-2}} \int_0^\infty \cdots \int_0^\infty \Phi\left(\frac{r_1 t}{|r'|}, \ldots, \frac{r_{n-1}t}{|r'|}, \frac{t}{|r'|} \right) \frac{r_1 \cdots r_{n-1}}{|r'|^{2n}}\, dr_1 \cdots dr_{n-1} - \ln|f(0)|, \end{aligned} \tag{4.2.32}$$

where $r' = (r_1, \cdots, r_{n-1}, 1)$ and

[20] Recall that we have set $\lambda = (\lambda^*, 1)$, where $\lambda^* = (\lambda_1, \cdots, \lambda_{n-1})$.

$$\Phi(r) = \Phi(r_1, \cdots, r_n) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \ln|f(r_1 e^{i\psi_1}, \cdots, r_n e^{i\psi_n})| d_n\psi.$$

The function $\Phi(r)$ is monotone nondecreasing in each of the variables r_i, since it is the average, over the distinguished boundary of the polycylinder $|z_i| \leqq r_i$, $i = 1, \cdots, n$, of the plurisubharmonic function $\ln|f(z)|$. Therefore

$$\Phi\left(\frac{r_1 t}{|r'|}, \cdots, \frac{r_{n-1} t}{|r'|}, \frac{t}{|r'|}\right) \leqq \Phi\left(\frac{\tau t}{\tau_0}\right), \tag{4.2.33}$$

where $\tau_0 = \min_i \tau_i$.

Note, moreover, that by (4.2.31) and the definition of the function $\Phi(r)$ we have

$$\Phi(\tau t/\tau_0) = N_{f,\tau}(t|\tau|/\tau_0) + \ln|f(0)|.$$

Hence, in view of (4.2.32), (4.2.33) and the equality $W_{2n-2} = \pi^{n-1}/(n-1)!$, we conclude that

$$N_f(t) \leqq \frac{(2\pi)^{n-1}}{W_{2n-2}} \Phi\left(\frac{\tau t}{\tau_0}\right) \int_0^{\infty} \cdots \int_0^{\infty} \frac{r_1 \cdots r_{n-1}}{(1 + \sum_{i=1}^{n-1} r_i^2)^n} dr_1 \cdots dr_{n-1} - \ln|f(0)|$$
$$= N_{f,\tau}\left(\frac{t|\tau|}{\tau_0}\right).$$

We have thus derived the estimates

$$N_f(\tau_0 t/|\tau|) \leqq N_{f,\tau}(t) \leqq C_\delta N_f(\delta t), \tag{4.2.34}$$

which imply that the functions $N_f(t)$ and $N_{f,\tau}(t)$ have the same order and type and belong to the same convergence class. Q.E.D.

4. *The functions $n_f(r_1, \cdots, r_n)$ and $N_f(r_1, \cdots, r_n)$.* In the foregoing discussion, the vector τ was kept fixed throughout. For a more delicate characterization of the distribution of the points of the manifold χ_f, it is natural to treat $n_{f,\tau}(t)$ and $N_{f,\tau}(t)$ as functions not only of t but also of τ. When this is done, the obvious equalities

$$n_{f,\alpha\tau}(t) = n_{f,\tau}(t), \qquad N_{f,\alpha\tau}(t) = N_{f,\tau}(t),$$

which hold for any $\alpha > 0$, imply that $n_{f,\tau}(t)$ and $N_{f,\tau}(t)$ are in fact functions not of the $n+1$ independent variables t and $\tau_1, \cdots, \tau_n$, but of the n variables $\tau_1 t, \cdots, \tau_n t$, where the numbers $\tau_1, \cdots, \tau_n$ satisfy the condition $\sum_1^n \tau_j^2 = 1$. Denote $\tau_1 t = r_1$, $\cdots, \tau_n t = r_n$, $r = (r_1, \cdots, r_n)$, and

$$n_f(r) = n_{f,\tau}(t), \qquad N_f(r) = N_{f,\tau}(t).$$

Note that the only points $z \in \chi_f$ affecting the values of the functions $n_f(r)$ and $N_f(r)$ are those whose coordinates satisfy the conditions $|z_1|/r_1 = \cdots = |z_n|/r_n$, and $|z| < |r|$, i.e. the points $(z_1, \cdots, z_n) \in \chi_f$ such that the corresponding points $(|z_1|, \cdots, |z_n|)$ in real n-space R^n lie on the segment joining the origin to the point $(r_1, \cdots, r_n)$.

In order to investigate the growth of the functions $N_f(r)$ and $n_f(r)$, we write the equality (4.2.31) proved above in terms of the function $N_f(r)$, this equality is obviously valid not only for entire functions but also for functions analytic in suitable polycylinders.

This results in the following theorem:

THEOREM 4.2.5 (JENSEN'S FORMULA). *Let the function $f(z)$ be holomorphic in the polydisk $\{z\colon |z_i| \leqq r_i, i = 1, \cdots, n\}$, and let $f(0) \neq 0$. Then*

$$N_f(r) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \ln\left|f(r_1 e^{i\varphi_1}, \cdots, r_n e^{i\varphi_n})\right| d_n\varphi - \ln\left|f(0)\right|. \tag{4.3.35}$$

The first term on the right of (4.2.35) is simply the average of the plurisubharmonic function $\ln|f(z)|$. It follows that the function $N_f(r)$ belongs to the class $\mathfrak{A}$ studied in § 6 of Chapter 2, and hence all the assertions proved there concerning the growth of functions in class $\mathfrak{A}$ are valid here. However, the function $n_f(r)$ is in general not in class $\mathfrak{A}$. Nevertheless, all assertions concerning orders of functions in $\mathfrak{A}$ extend to this function. Indeed, it follows from the definitions of $N_f(r)$ and $n_f(r)$ that for any $\delta > 1$, some $R_0 > 0$, and all $r \in \{r\colon r \in R_+^n, |r| > R_0\}$ we have the inequalities

$$\frac{1}{\ln|r/R_0|} N_f(r) \leqq n_f(r) \leqq \frac{1}{\ln \delta} N_f(\delta r).$$

These inequalities imply that the orders of the function $N_f(r)$, by any of the definitions considered in § 6 of Chapter 2, coincide with the similarly defined orders of $n_f(r)$. Moreover, the characterization given in Theorem 2.6.1 for the hypersurface of associated orders remains exact even when $\mathfrak{A}$ is replaced by the subclass consisting of the functions $N_f(r)$. This will follow from the following theorem.

THEOREM 4.2.6 (RONKIN [12]). *Let B be a complete convex domain in R_+^n. Let B^* be the image of B under the mapping $a_i = 1/b_i$, $i = 1, \cdots, n$. Then there exists an entire function $f(z)$ such that $\bar{B}^* = \bar{B}_\rho(N_f)$, where $B_\rho(N_f)$ is the set figuring in the definition of the hypersurface of associated orders of the function $N_f(r)$.*

PROOF. We first observe that, by the convexity and completeness of the set B, the set B^* can be expressed as

$$B^* = \bigcap_1^\infty B_m, \tag{4.2.36}$$

where

$$B_m = \left\{a\colon a \in R_+^n, \sum_{i=1}^n \frac{\alpha_i^{(m)}}{a_i} \leqq 1\right\}, \qquad \alpha_i^{(m)} > 0, \quad i = 1, \cdots, n.$$

Without loss of generality we may assume that $\alpha_i^{(m)} = p_i^{(m)}/q_m$, where $p_i^{(m)}$ and q_m

are integers. Now let $\psi_m(w)$ be entire functions of one variable such that $\rho(N_{\psi_m}) = 1/q_m$, $\psi_m(0) = 1$ and for any $R > 0$

$$N_{\psi_m}(R) \leqq R^{1/q_m}.$$

It is easy to see that for this choice of functions $\psi_m(w)$ the hypersurface of associated orders of the function $N_{f_m}(r)$, where

$$f_m(z) = \psi_m(z_1^{p_1^{(m)}} \cdots z_n^{p_n^{(m)}}),$$

is the hypersurface

$$\sum_{i=1}^{n} \frac{\alpha_i^{(m)}}{a_i} = 1.$$

Indeed, by Jensen's formula

$$N_{f_m}(r) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \ln \left| \psi_m\left(r_1^{p_1^{(m)}} \cdots r_n^{p_n^{(m)}} \exp\left(i \sum_1^n \varphi_j p_j^{(m)}\right)\right)\right| d_n\varphi$$
$$= N_{\psi_m}\left(r_1^{p_1^{(m)}} \cdots r_n^{p_n^{(m)}}\right),$$

and thus, since $N_{\psi_m}(R) \leqq R^{1/q_m}$, for all $r_1 > 0, \cdots, r_n > 0$ it follows that

$$N_{f_m}(r_1, \cdots, r_n) \leqq \prod_{j=1}^{n} r_j^{p_j^m/q_m} = \prod_{j=1}^{n} r_j^{\alpha_j^{(m)}}.$$

Further, in view of the obvious inequality

$$\prod_{j=1}^{n} r_j^{\alpha_j^{(m)}} \leqq \left(\sum_{j=1}^{n} r_j^{a_j}\right)^{\sum_1^n \alpha_j^{(m)}/a_j}$$

we see that on the hypersurface $\sum_1^n \alpha_j^{(m)}/a_j = 1$, for all $r_1 > 0, \cdots, r_n > 0$,

$$N_{f_m}(r) \leqq \sum_{j=1}^{n} r_j^{a_j}.$$

On the other hand, the inequality

$$N_{f_m}(r) \leqq C_\varepsilon + \sum_{j=1}^{n} r_j^{a_j}$$

cannot hold for all $r_1 > 0, \cdots, r_n > 0$ for any $\varepsilon > 0$; otherwise, setting $r_j = R^{1/a_j q_m}$, $j = 1, \cdots, n$, we could deduce that for any $R > 0$

$$N_{f_m}(R^{1/a_1 q_m}, \cdots, R^{1/a_n q_m}) = N_{\psi_m}(R) \leqq C_\varepsilon + \sum_{i=1}^{n} R^{1/q_m - \varepsilon/a_i q_m}.$$

This contradicts the equality $\rho(N_{\psi_m}) = 1/q_m$. Thus the numbers $a_1, \cdots, a_n$ such that $\sum_1^n \alpha_i^{(m)}/a_i = 1$ form a system of associated orders for the function $N_{f_m}(r)$.

We now take a sequence of positive numbers $\lambda_1, \lambda_2, \cdots$ which approach zero so rapidly that the infinite product

$$f(z) = \prod_{m=1}^{\infty} f_m(\lambda_m z)$$

is absolutely convergent in every bounded region of C^n. This is always possible, since $f_m(0) = 1$ for any m. We also require that for any $\alpha > 0$

$$\sum_{m=1}^{\infty} \lambda_m^{\alpha} < \infty. \tag{4.2.37}$$

By virtue of the absolute convergence of the product $f(z)$, it follows immediately from Jensen's formula that

$$N_f(r) = \sum_{m=1}^{\infty} N_{f_m}(\lambda_m r). \tag{4.2.38}$$

For any m and all $r_1 > 0, \cdots, r_n > 0$, we have $N_{f_m}(r) \geqq 0$. Consequently $N_{f_m}(r) \leqq N_f(r)$ for any m. Hence, since the factor λ_m does not affect the hypersurface of associated orders, we conclude that $\bar{B}_\rho(N_f) \subset B_m$, and so

$$\bar{B}_\rho(N_f) \subset \bigcap_{m=1}^{\infty} B_m. \tag{4.2.39}$$

To prove the reverse inclusion, suppose that

$$(a_1, \cdots, a_n) \in \bigcap_{m=1}^{\infty} B_m.$$

Then for any m and all $r_1 > 0, \cdots, r_n > 0$ we have

$$N_{f_m}(r) \leqq \prod_{i=1}^{n} (\lambda_m r_i)^{\alpha_i^{(m)}} \leqq \lambda_m^{\sum_1^n \alpha_i^{(m)}} \left(\sum_{i=1}^{n} r_i^{a_i} \right)^{\sum_1^n \alpha_i^{(m)}/a_i} \leqq \lambda_m^{\sum_1^n \alpha_i^{(m)}} \sum_{i=1}^{n} r_i^{a_i}. \tag{4.2.40}$$

We now observe that, owing to the properties of the domain B,

$$\inf_m \sum_{i=1}^{n} \alpha_i^{(m)} > 0.$$

From this and (4.2.37) it follows that

$$\sum_{m=1}^{\infty} \lambda_m^{\sum_1^n \alpha_i^{(m)}} < \infty. \tag{4.2.41}$$

Further, we see from (4.2.38), (4.2.40) and (4.2.41) that for any $r_1, \cdots, r_n$

$$N_f(r) \leqq C \sum_{i=1}^{n} r_i^{a_i},$$

where C is a constant. Consequently $(a_1, \cdots, a_n) \in \bar{B}_\rho(N_f)$, and hence

$$\bigcap_{m=1}^{\infty} B_m \subset \bar{B}_\rho(N_f).$$

Together with (4.2.39), this implies that

$$\bar{B}_\rho(N_f) = \bigcap_{m=1}^{\infty} B_m,$$

and finally, by virtue of (4.2.36), we get $\bar{B}_\rho(N_f) = B^*$. Q.E.D.

5. *The functions $N_{f,z_n}(t, z^*)$ and $n_{f,z_n}(t, z^*)$.* In the final analysis, the functions $n_f(t)$, $N_f(t)$, etc. considered in the preceding subsections of this section were based on the function $n_f(t; \kappa)$, which characterizes the distribution of the points of χ_f on analytic planes through the origin. In order to describe the distribution of points of χ_f in the direction of one of the coordinate planes, say the z_n-plane, a natural substitute for $n_f(t; \kappa)$ is the function $n_{f,z_n}(t, z_1, \cdots, z_{n-1})$ defined as the number of zeros of the function $f(z_1, \cdots, z_n)$ in the disk $|z_n| \leqq t$ for fixed $z_1, \cdots, z_{n-1}$.

To abbreviate the notation both here and in the next section, if $a = (a_1, \cdots, a_n)$ is a complex or real vector, we shall denote the vector $(a_1, \cdots, a_{n-1})$ by a^*.[21] Thus we shall write $f(z) = f(z^*, z_n)$, $n_{f,z_n}(t, z_1, \cdots, z_{n-1}) = n_{f,z_n}(t, z^*)$ and so on.

It follows from the definition of the function $n_{f,z_n}(t, z^*)$ that it characterizes the distribution of the points of χ_f on every complex one-dimensional analytic plane parallel to the z_n-plane. The distribution of points of χ_f on cylinders of the type $\{z: |z_i| = r_i, i = 1, \cdots, n-1\}$ has a natural characterization in terms of the function

$$n_f(t; r^*) = n_{f,z_n}(t; r^*) = \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} n_{f,z_n}(t, r_1 e^{i\varphi_1}, \cdots, r_{n-1} e^{i\varphi_{n-1}}) d_{n-1}\varphi.$$

We shall also consider the functions

$$N_{f,z_n}(t, z^*) = \int_0^t \frac{n_{f,z_n}(s, z^*) - n_{f,z_n}(0, z^*)}{s} ds + n_{f,z_n}(0, z^*) \ln t$$

and

$$N_f(t; r^*) = N_{f,z_n}(t; r^*) = \int_0^t \frac{n_f(s; r^*)}{s} ds.^{22}$$

Note that with respect to the variable t the functions $N_{f,z_n}(t, z^*)$ and $n_{f,z_n}(t, z^*)$ have the same order, type (minimal, normal or maximal) and belong to the same convergence class. The analogous assertion is of course true for the functions $n_f(t; r^*)$ and $N_f(t; r^*)$. On the other hand, as in the case of $n_{f,\tau}$ and $N_{f,\tau}$, the function $N_{f,z_n}(t, z^*)$ is easier to handle than $n_{f,z_n}(t, z^*)$, since it has a comparatively simple analytic expression. Indeed, suppose that $f(z^*, z_n) \not\equiv 0$ for fixed z^*; then, by Jensen's formula for analytic functions of one variable,

[21] This notation was used in the preceding sections, but not systematically.

[22] By (c) of §1.6 we have $n_f(0; r^*) = 0$.

$$N_{f,z_n}(t, z^*) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(z^*, te^{i\varphi_n})\right| d\varphi_n + \alpha(z^*), \tag{4.2.42}$$

where $\alpha(z^*) = -\ln\left|f(z^*, 0)\right|$ if $f(z^*, 0) \neq 0$, and

$$\alpha(z^*) = -\ln\left|\frac{1}{l!}\frac{\partial^l f}{\partial z_n^l}\right|_{z_n=0}$$

if $z_n = 0$ is a zero of multiplicity l of the function $f(z^*, z_n)$. The corresponding formula for the function $N_{f,z_n}(t; r^*)$ has the form

$$N_{f,z_n}(t; r^*) = N_f(r^*, t) - N_f(r^*, 0). \tag{4.2.43}$$

Here the function $N_f(r^*, 0)$ is defined by (4.2.35), and it is assumed that $f(z^*, 0) \not\equiv 0$.

Indeed, since the Lebesgue measure of the set $\chi_j \cap \{z: |z_i| = r_i \text{ for } i = 1, \cdots, n-1; z_n = 0\}$ is zero, it follows from (4.2.35) and (4.2.42) that

$$\begin{aligned}
N_{f,z_n}(t; r^*) &= \int_0^t \frac{n_f(s; r^*)}{s}\, ds \\
&= \frac{1}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi} N_{f,z_n}(t, r_1e^{i\varphi_1}, \cdots, r_{n-1}e^{i\varphi_{n-1}})\, d_{n-1}\varphi \\
&= \left(\frac{1}{2\pi}\right)^{n-1}\int_0^{2\pi}\cdots\int_0^{2\pi} d_{n-1}\varphi \left\{\frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(r_1e^{i\varphi_1}, \cdots, r_{n-1}e^{i\varphi_{n-1}}, te^{i\varphi_n})\right| d\varphi_n \right. \\
&\qquad \left. + \alpha(r_1e^{i\varphi_1}, \cdots, r_{n-1}e^{i\varphi_{n-1}})\right\} \\
&\quad - \left(\frac{1}{2\pi}\right)^{n-1}\int_0^{2\pi}\cdots\int_0^{2\pi} \ln\left|f(r_1e^{i\varphi_1}, \cdots, r_{n-1}e^{i\varphi_{n-1}}, 0)\right| d_{n-1}\varphi \\
&= \left(\frac{1}{2\pi}\right)^{n}\int_0^{2\pi}\cdots\int_0^{2\pi} \ln\left|f(r_1e^{i\varphi_1}, \cdots, r_{n-1}e^{i\varphi_{n-1}}, te^{i\varphi_n})\right| d_n\varphi \\
&= N_f(r^*, t) - N_f(r^*, 0).
\end{aligned}$$

Despite its simple proof, equality (4.2.43) seems to us quite surprising, since it establishes a connection between the distribution of the zeros of $f(z)$ on the cone

$$\{z: |z_1|/r_1 = \cdots = |z_n|/r_n, |z| \leqq |r|\}$$

and on the cylinder

$$\{z: |z_1| = r_1, \cdots, |z_{n-1}| = r_{n-1}, |z_n| \leqq r_n\}.$$

In particular, it follows from (4.2.43) that all questions relating to the growth of the function $N_{f,z_n}(t; r^*)$ in the variable t reduce to the corresponding questions for the growth of $N_f(r)$ in the variable r_n, and this latter function is in the class $\mathfrak{A}$. Thus all propositions proved in § 6 of Chapter 2 on the growth in r_n of a function $\Phi(r) \in \mathfrak{A}$ are also valid for the function $N_{f,z_n}(r_n; r^*)$.

Now the first term on the right of (4.2.42) was shown in § 7 of Chapter 2 to belong to class $\mathfrak{B}$. All propositions proved there concerning the growth of a function $\Phi(z, t) \in \mathfrak{B}$ in the variable t are therefore valid for the function $N_{f,z_n}(t, z^*)$.

In general, the functions $n_{f,z_n}(t; r^*)$ and $n_{f,z_n}(t, z^*)$ need not belong to the classes $\mathfrak{A}$ and $\mathfrak{B}$, respectively. However, because of the relationship of these functions to $N_{f,z_n}(t; r^*)$ and $N_{f,z_n}(t, z^*)$, it is clear that they have the same properties as functions of these classes in regard to order, type (minimal, normal or maximal) and convergence class (with respect to the variable t). In particular, we have the following theorem:

THEOREM 4.2.7. *Let $E \subset C^{n-1}$ be a set of positive Γ-capacity on which the order $\rho(z^*; n_{f,z_n}(t, z^*))$ of the function $n_{f,z_n}(t, z^*)$ in the variable t satisfies the condition*

$$\rho(z^*; n_{f,z_n}(t, z^*)) < \infty.$$

Then the order $\bar{\rho}$ of the function $n_{f,z_n}(t; r^*)$[23] in the variable t is finite, and the equality

$$\rho(z^*; n_{f,z_n}(t, z^*)) = \bar{\rho}$$

holds throughout C^{n-1}, with the possible exception of a set K_f which is an F_σ set satisfying the following condition:

(A) $K_f \neq C^{n-1}$, and the intersection of K_f with any analytic plane $\{z^*: z_i = a_i w + b_i, w \in C^1, i = 1, \cdots, n-1\}$ not a subset of K_f has zero capacity.[24]

The function $N_{f,z_n}(t, z^*)$ may be estimated from above in terms of $N_{f,z_n}(t; r^*)$. When this is done, we get the following theorem, which is similar to Theorem 4.2.3.

THEOREM 4.2.8. *Let $f(z)$ be an entire function in C^n, $f(0) \neq 0$ and $f(z^*, z_n) \not\equiv 0$ for all z^*.*[25] *Then for any $\delta > 1$ there exist a constant $C_\delta > 0$ and a function $\beta_{\delta,f}(r^*)$, bounded on every compact subset of R_+^n, such that for any $t > 0$*

$$N_{f,z_n}(t, z^*) \leqq C_\delta N_{f,z_n}(t; \delta|z_1|, \cdots, \delta|z_{n-1}|) + \beta_{\delta,f}(|z_1|, \cdots, |z_{n-1}|) + \alpha(z^*), \tag{4.2.44}$$

where $\alpha(z^*)$ is the same as in (4.2.42).

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[23] The order of the function $n_f(t; r^*)$ in the variable t for fixed $r_1 > 0, \cdots, r_{n-1} > 0$ is independent of $r_1, \cdots, r_{n-1}$.

[24] The question of the behavior of $n_{f,z_n}(t, z^*)$ as a function of z^* is an old one. In 1926, Julia [1] posed the question of the "richness" of the set of points z^* for which $n_{f,z_n}(t, z^*) \equiv 0$. Under certain natural restrictions, he proved that in the two-variable case this set cannot contain a continuum. In 1941, Lelong [1] showed, again in the two-variable case, that the set in question has inner capacity zero. In the same paper, Lelong also proved that the set of points $z_1 \in C^1$ for which the order of $n_{f,z_2}(t, z_1)$ differs from that of $N_{f,z_2}(t; r_1)$ has no interior points. The study of these questions was more difficult than the analogous questions for the function $M_f(z, t)$, owing to the impossibility of using power series. These difficulties are overcome here thanks to our general approach to this type of problem, embodied in the treatment of $N_{f,z_n}(t, z^*)$ and $\ln M_f(z, t)$ as functions of class $\mathfrak{B}$.

[25] Note that any entire function $f(z)$ can be expressed as $f(z) = z_n^l g(z^*)\phi(z)$, where l is a nonnegative integer, $g(z^*)$ an entire function in C^{n-1}, and $\phi(z)$ an entire function such that $\dim\{z^*: \phi(z^*, z_n) = 0 \ \forall z_n \in C^1\} \leqq n - 3$.

PROOF. Since $f(z^*, z_n) \not\equiv 0$ for any fixed z^*, it is readily seen that the function

$$\Phi(z^*, t) = \int_0^{2\pi} \ln|f(z^*, te^{i\varphi_n})|d\varphi_n$$

satisfies the condition

$$\inf_{z\in G, t\geqq 1} \Phi(z^*, t) > -\infty$$

in every bounded domain $G \subset C^{n-1}$. Without loss of generality we may assume that $\Phi(z_1', \cdots, z_{n-1}', t) \geqq 0$ for $|z_i'| \leqq \delta|z_i|$, $i = 1, \cdots, n-1$. Taking this into consideration, let us estimate the function $\Phi(z^*, t)$, which is plurisubharmonic in C^{n-1}, using the properties of its average $\mathfrak{M}_\Phi^\varepsilon$. Setting $\delta = 1 + \varepsilon$, we get

$$\begin{aligned}
\Phi(z^*, t) &\leqq \frac{1}{(\pi\varepsilon^2)^{n-1}|z_1|^2 \cdots |z_{n-1}|^2} \int\limits_{\substack{|z_i - z_i'| < \varepsilon|z_i| \\ i=1,\cdots,n-1}} \Phi(z'^*, t)\, d\omega_{z'^*} \\
&\leqq \frac{1}{(\pi\varepsilon^2)^{n-1}|z_1|^2 \cdots |z_{n-1}|^2} \int\limits_{\substack{|z_i'| \leqq \delta|z_i| \\ i=1,\cdots,n-1}} \Phi(z^*, t)\, d\omega_{z'^*} \\
&\leqq \frac{\prod_{i=1}^{n-1} \delta^2|z_i|^2/2}{(\pi\varepsilon^2)^{n-1}\prod_{i=1}^{n-1}|z_i|^2} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi(\delta|z_1|e^{i\varphi_1}, \cdots, \delta|z_{n-1}|e^{i\varphi_{n-1}}, t)\, d_{n-1}\varphi \\
&= C_\delta \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln|f(\delta|z_1|e^{i\varphi_1}, \cdots, \delta|z_{n-1}|e^{i\varphi_{n-1}}, te^{i\varphi_n})|d_n\varphi \\
&= C_\delta N_f(\delta|z_1|, \cdots, \delta|z_{n-1}|, t) + C_\delta \ln|f(0)|,
\end{aligned}$$

where

$$C_\delta = 2\pi\delta^{2n-2}/(\delta - 1)^{2n-2}.$$

Hence by (4.2.42) and (4.2.43) we get the required inequality (4.2.44). Q.E.D.

§ 3. On analogs of the Weierstrass canonical product

1. *Statement of the problem.* One of the basic tools for studying entire functions of finite order in the one-variable case is representation of these functions in the form

$$f(z) = e^{P(z)}\varphi(z), \tag{4.3.1}$$

where $P(z)$ is a polynomial, and the function $\varphi(z)$, the Weierstrass canonical product, has the form

$$\varphi(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z}{a_i}\right)\exp\left(\frac{z}{a_i} + \frac{z^2}{2a_i^2} + \cdots + \frac{z^q}{qa_i^q}\right). \tag{4.3.2}$$

Here q is the smallest integer such that the integral $\int^\infty n_f(t)t^{-q-2}\,dt$ converges. As everywhere in this section, it is assumed that $f(0) \neq 0$.

It is clear that this representation cannot carry over directly to the case of entire functions of several variables, since here the zeros do not constitute a discrete set. Nevertheless, the problem of finding analogs of (4.3.1) can be formulated for $n > 1$, in view of the fact that the function (φz) in (4.3.2) is uniquely determined by the following of its properties:

(1) *The function* $f\varphi^{-1}$ *is an entire function which vanishes nowhere.*

(2) *The first term in the power series expansion of* $\ln \varphi(z)$ *in the neighborhood of* $z = 0$ *is* $c_{q+1} z^{q+1}$.

(3) *Of all functions possessing properties* (1) *and* (2), *the function* $\varphi(z)$ *is that with minimal growth, in the sense that*

$$\ln M_\varphi(R) \leqq \text{const} \cdot K_q(R; n_f),$$

where, as before,

$$K_q(R; n_f) = R^q \left\{ \int_0^R \frac{n_f(s)}{s^{q+1}} ds + R \int_R^\infty \frac{n_f(s)}{s^{q+2}} ds \right\}.$$

With this characterization of the canonical function $\varphi(z)$, the differences between the one-variable and several-variable cases disappear, and the problem of finding a representation (4.3.1) becomes meaningful for entire functions of n variables as well. The problem was solved by various methods by Lelong [4], [10] and Stoll [1]. Later, in a formulation slightly different from that adopted by Stoll and Lelong, it was solved by the present author [12]. All the above-mentioned authors made use of the following idea, which goes back to Kneser [1]:

It follows from (4.3.2) that

$$\ln \varphi(z) = \sum_{i=1}^\infty \left\{ \frac{z}{a_i} + \frac{z^2}{2a_i^2} + \cdots + \frac{z^q}{qa_i^q} + \ln\left(1 - \frac{z}{a_i}\right) \right\}.$$

The sum on the right may be regarded, in a certain sense, as the zero-dimensional integral of the function

$$\sum_{m=1}^q \frac{z^m}{ma^m} + \ln\left(1 - \frac{z}{a}\right)$$

over the set of zeros of the function $\varphi(z)$, or, equivalently, of the function $f(z)$. The required analog of (4.3.2) will therefore be a representation in which the logarithm of an entire function of several variables is expressed in terms of an integral over its zero surface.

2. *Canonical function.* Lelong's solution of the problem formulated in § 3.1 is contained in the following theorem.

THEOREM 4.3.1 (LELONG [4], [10]). *Let* $f(z)$, $z \in C^n$, *be an entire function such that* $f(0) \neq 0$ *and the function* $\nu_f(t)$[26] *corresponding to the function* $\ln|f|$ *has finite order. Let* q *be the smallest integer such that the integral*

[26] The notation is as before; see §§2 and 3 of Chapter 1 and § 4 of Chapter 2.

$$\int^{\infty} \frac{\nu_f(t)}{t^{q+2}}\, dt$$

is convergent. Then the function $f(z)$ *has a unique representation*

$$f(z) = \varphi(z)e^{g(z)},$$

where the entire functions $\varphi(z)$ *and* $g(z)$ *satisfy the following conditions*:

(1) *In the neighborhood of the origin, the function* $\ln \varphi(z)$ *can be expanded in a power series as follows*:

$$\ln \varphi(z) = \sum_{\|k\|=q+1}^{\infty} a_k z^k.$$

(2) *For some constant* $C = C(n, q)$, *the following estimate holds everywhere on the half-axis* $R > 0$:

$$\ln M_\varphi(R) \leqq C K_q(R; \nu_f).^{27)} \tag{4.3.3}$$

When these conditions hold,

$$\ln |\varphi(z)| = J_\mu(z), \tag{4.3.4}$$

$$\ln \varphi(z) = 2 \int_0^z \sum_{i=1}^{n} \frac{\partial J_\mu}{\partial z_i}\, dz_i, \tag{4.3.5}$$

where $J_\mu(z)$ *is the canonical potential of the Riesz measure* μ *associated with the function* $\ln |f|$, *and the path of integration in* (4.3.5) *lies within the set* χ_f (*and is otherwise arbitrary*).

PROOF. By Theorem 2.4.1 the function $\ln |f|$, which is plurisubharmonic in C^n, admits the representation

$$\ln |f(z)| = J_\mu(z) + \Phi(z), \tag{4.3.6}$$

where J_μ is the canonical potential mentioned in the statement of the theorem, and $\Phi(z)$ a pluriharmonic function. Let $g(z)$ be an entire function such that $\Phi(z) = \operatorname{Re} g(z)$ — such a function exists, for $\Phi(z)$ is pluriharmonic. It then follows from (4.3.6) that

$$J_\mu(z) = \ln|f(z)| - \operatorname{Re} g(z) = \ln|\varphi(z)|, \tag{4.3.7}$$

where the entire function $\varphi(z)$ is defined by $\varphi(z) = f(z)e^{-g(z)}$. We claim that the resulting representation $f(z) = \varphi(z)e^{g(z)}$ is the required one. Indeed, it follows immediately from the estimate given in § 3 of Chapter 1 for the canonical potential, and from (4.3.7), that $\varphi(z)$ satisfies condition (2) of the theorem. Further, as was shown in § 3 of Chapter 2, all derivatives of the canonical potential $J_\mu(z)$ of order $\leqq q$ vanish at the origin. Hence, since $\ln \varphi(z) = J_\mu(z) + i \operatorname{Im} \ln \varphi(z)$ is analytic in

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27) Here $M_\varphi(R) = \max_{|r| \leq R} M_\varphi(r_1, \cdots, r_n)$.

the neighborhood of the origin, it follows that all derivatives of $\ln \varphi(z)$ of order $\leqq q$ also vanish at $z = 0$, and so $\varphi(z)$ satisfies condition (1). Finally, the equality $f(z) = \varphi(z)e^{g(z)}$ holds by virtue of the definition of $\varphi(z)$, while condition (4.3.5) is simply the well-known rule for construction of an analytic function from its real part. Thus in order to complete the proof it remains to show that the representation $f = \varphi e^g$ is unique. Suppose that a similar representation is valid with other functions φ_1 and g_1. Then the function

$$\psi(z) = \varphi(z)/\varphi_1(z)$$

is entire, and moreover $\psi(z) \neq 0$ for all $z \in C^n$, and

$$\ln \psi(z) = \sum_{\|k\|=q+1} b_k z^k. \tag{4.3.8}$$

On the other hand, it follows from Lemma 1.3.2 that $\ln|\psi(z)|$ increases no faster than $\max[\ln|\varphi|, \ln|\varphi_1|]$. Hence in view of (4.3.3) we conclude that $\psi(z)$ is of at most minimal type for order $q + 1$, and so

$$\psi(z) = e^{P_q(z)}, \tag{4.3.9}$$

where $P_q(z)$ is a polynomial of degree at most q. It follows from (4.3.9) and (4.3.8) that $\psi(z) = 1$, and hence $\varphi_1(z) = \varphi(z)$ and $g_1(z) = g(z)$. Q.E.D.

REMARK 1. The estimates of the function $\varphi(z)$ given in Theorem 4.3.1 imply that its order is the same as that of the function $\nu_f(t)$, and when this order is $q + 1$ the function $\varphi(z)$ is of minimal type.

REMARK 2. It is readily seen that the functions figuring in the representation $f = \varphi e^g$ remain unchanged if instead of condition (2) of Theorem 4.3.1 we require that the function $\varphi(z)$ increase no faster than a function of minimal type for order $q + 1$.

The conditions imposed on the set χ_f under which Theorem 4.3.1 was proved may be replaced by conditions on the set $\chi_{f,\tau} = \{z: f(z) = 0, |z_1|/\tau_1 = \cdots = |z_n|/\tau_n\}$. This follows from Theorems 4.2.2 and 4.2.4. We get the following theorem.

THEOREM 4.3.2 (RONKIN [12]). *Let $f(z)$ be an entire function, $f(0) \neq 0$, and let the function $n_{f,\tau}(t)$ have finite order for some τ. Let q be the smallest integer such that the integral*

$$\int^{\infty} \frac{n_{f,\tau}(t)}{t^{q+2}}\, dt$$

converges. Then there exists an entire function $\varphi(z)$ satisfying the following conditions:

(1) *The function $f(z)\varphi^{-1}(z)$ is an entire function in C^n and does not vanish.*

(2) *The expansion of the function* $\ln \varphi(z)$ *in the neighborhood of the origin has the form*

$$\ln \varphi(z) = \sum_{\|k\|=q+1} a_k z^k.$$

(3) *For any* $R > 0$,

$$\ln M_\varphi(R) \leqq C_{n,q} K_q(R; n_f). \tag{4.3.10}$$

Instead of (4.3.10), we can prove the inequality

$$\ln M_\varphi(R) \leqq CK(R; N_{f,\tau}(\delta t)),$$

where δ is any number greater than $|\tau|\{\min_i \tau_i\}^{-1}$ and $C = C(\delta, n, q)$ a constant. This follows from (4.2.34) and (4.3.10). This form of condition (3) is useful when only characteristics of the set $\chi_{f,\tau}$ are considered.

When $n = 1$, the function $\varphi(z)$, which by Remark 2 is the same in both theorems, is the Weierstrass canonical product based on the zeros of the function $f(z)$. When $n > 1$ we shall call it the canonical function corresponding to the manifold $\chi_f = \{z: f(z) = 0\}$ (counting multiplicities). It follows from the generalized Jensen formula (see § 2 of Chapter 1) and from the estimates for $\varphi(z)$ in Theorem 4.3.1 that any function $\psi(z)$ for which $\chi_\psi = \chi_f$ (counting multiplicities) increases no slower than $\varphi(z)$.

The existence of the canonical function implies an extremely useful property of the set χ_f. This is presented in the next theorem.

THEOREM 4.3.3 (RONKIN [12]). *Let $f(z)$ be an entire function, $f(0) \neq 0$, such that for some τ the function $n_{f,\tau}(t)$ has finite order ρ. Let $w_j(\lambda)$, $\lambda \in C^n$, denote the zeros of the function $f(\lambda w)$, regarded as a function of the single variable w for fixed $\lambda_1, \cdots, \lambda_n$. Then if $m > q$*[28] *the series*

$$\sum_{j=1}^{\infty} w_j^{-m}(\lambda) \tag{4.3.11}$$

is absolutely convergent and its sum is a homogeneous polynomial of degree m in $\lambda_1, \cdots, \lambda_n$.

PROOF. By Theorem 4.3.2 the numbers $w_j(\lambda)$ may be defined not as zeros of the function $f(\lambda w)$ but as the zeros of $\varphi(\lambda w)$, where $\varphi(z)$ is the canonical function of the set χ_f. It follows from the properties of the canonical function that $\varphi(\lambda w)$ is of order at most ρ in w. On the other hand, by Theorem 4.2.4

$$\int^{\infty} \frac{n_f(t)}{t^{q+2}}\, dt < \infty,$$

whence by Theorem 4.2.3 it follows that for any $\kappa \in P_{n-1}$

$$\int^{\infty} \frac{n_f(t; \kappa)}{t^{q+2}}\, dt < \infty$$

and, moreover,

$$\int^{\infty} \frac{dn_f(t; \kappa)}{t^{q+1}} < \infty.$$

[28] q is the same as in Theorem 4.3.2.

Therefore, for any $\lambda_1, \cdots, \lambda_n$ and any $m \geqq q + 1$,

$$\sum_{j=1}^{\infty} |w_j(\lambda)|^{-m} < \infty,$$

and the function $\varphi(\lambda w)$ has the representation

$$\varphi(\lambda w) = e^{P_\lambda(w)} \prod_{j=1}^{\infty} \left(1 - \frac{w}{w_j}\right) \exp\left(\sum_{p=1}^{q} \frac{w^p}{p w_j^p}\right),$$

where $w_j = w_j(\lambda)$ and $P_\lambda(w)$ is a polynomial of degree at most q with coefficients depending on λ. Now note that the expansions of the functions $\ln \varphi(\lambda w)$ and

$$\ln\left\{\prod_{j=1}^{\infty} \left(1 - \frac{w}{w_j}\right) \exp\left(\sum_{p=1}^{q} \frac{w^p}{p w_j^p}\right)\right\}$$

in powers of w contain no terms in w^m for $m \leqq q$. This immediately implies that $P_\lambda(w) \equiv 0$, and so

$$\varphi(\lambda w) = \prod_{j=1}^{\infty} \left(1 - \frac{w}{w_j}\right) \exp\left(\sum_{p=1}^{q} \frac{w^p}{p w_j^p}\right).$$

Consequently the sum $-m^{-1} \sum_1^{\infty} w_j^{-m}$ is the coefficient of w^m in the expansion of $\ln \varphi(\lambda w)$ in powers of w:

$$\frac{1}{m} \sum_{j=1}^{\infty} w_j^{-m} = - \sum_{\|k\|=m} \frac{\lambda^k}{k!} \frac{\partial^{\|k\|} \ln \varphi(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \bigg|_{z=0}.$$

This completes the proof.

It will follow from results of § 3.3 that in a certain sense the above property of the sum (4.3.11) is equivalent to the existence of the canonical function. However, we know of no proof of the Theorem 4.3.3 that does not depend on the existence of the canonical function.

3. *Construction of the canonical function from the set* $\chi_{f,\tau}$. We shall now derive an integral representation of the canonical function $\varphi(z)$ which involves not the entire zero manifold χ_f but only its $(n-1)$-dimensional submanifold $\chi_{f,\tau}$. Before formulating the rigorous result, we recall the notation

$$z/\zeta = (z_1/\zeta_1, \cdots, z_n/\zeta_n);$$

w will denote a complex (scalar) parameter, and $d_n\varphi$ the volume element in the space of the variables $\varphi_1, \cdots, \varphi_n$.

The following theorem is an analog of the theorems of Lelong and Stoll mentioned in § 3.2.

THEOREM 4.3.4 (RONKIN [12]). *Let $f(z)$ be an entire function such that $f(0) \neq 0$ and, for some fixed $\tau, |\tau| = 1$, the function $N_{f,\tau}(t)$ is of order $\rho(N_{f,\tau}) = \rho < \infty$. Then the following assertions are true.*

(1) *The following function exists and is an entire function:*

$$g(z) = \frac{2}{(2\pi)^n} \lim_{R\to\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln |f(\zeta')| F_q\left(\frac{z}{\zeta'}\right) d_n\varphi, \tag{4.3.12}$$

where

$$\zeta' = (R\tau_1 e^{i\varphi_1}, \cdots, R\tau_n e^{i\varphi_n}), \qquad F_q(z) = \sum_{\|k\|>q} z^k,$$

and q is the smallest integer for which the integral

$$\int^{\infty} \frac{N_{f,\tau}(t)}{t^{q+2}}\, dt$$

is convergent.

(2) *The logarithm of the function $\varphi^*(z) = f(z)e^{-g(z)}$ admits the following representation in some neighborhood of the origin:*[29]

$$\ln \varphi^*(z) = -\frac{1}{W_\tau} \int_{\chi_{f,\tau}} \gamma_f(\zeta) \Phi_q\left(\frac{z}{\zeta}\right) dW_\tau(\zeta) + P_q(z),$$

where

$$\Phi_q(z) = \sum_{\|k\|>q} \frac{z^k}{\|k\|},$$

and $P_q(z)$ is a polynomial of degree at most q.

(3) *The function $\varphi(z) = \varphi^*(z)e^{-P_q(z)}$ is the canonical function of the set χ_f.*

PROOF. It follows from the definition of q and from Theorems 4.2.3 and 4.2.4 that for any $\kappa \in P_\tau$

$$\int^{\infty} \frac{N_f(t;\kappa)}{t^{q+2}}\, dt < \infty.$$

Therefore, by well-known theorems from the theory of entire functions of one variable[30] the following representation holds for any $\kappa \in P_\tau$:

$$f(w\colon \kappa) = e^{h(w;\kappa)} \Pi(w;\kappa), \tag{4.3.13}$$

where

$$f(w;\kappa) = f(\tau_1 e^{i\psi_1} w, \cdots, \tau_{n-1} e^{i\psi_{n-1}} w),$$

$h(w;\kappa)$ is an entire function of w, and

$$\Pi(w;\kappa) = \prod_{j=1}^{\infty} \left(1 - \frac{w}{w_j}\right) \exp\left\{\sum_{m=1}^{q} \frac{1}{m}\left(\frac{w}{w_j}\right)^m\right\}.$$

[29] Here $\kappa \in P_{n-1}$ is defined by $\zeta \in C^n$, as follows: $\kappa = \{z\colon z_i = \zeta_i w, i = 1, \cdots, n, |w| < \infty\}$.

[30] See, for example, Levin [1].

Here $w_j = w_j(\kappa)$ are the zeros of the function $f(w; \kappa)$, indexed in order of increasing absolute value.

We now perform the substitution $\varphi_n = \phi$, $\varphi_i = \psi_i + \phi$, $i = 1, \cdots, n-1$, in the integral (4.3.12), and then use the expansion (4.3.13). This gives

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \ln|f(\zeta')| F_q\left(\frac{z}{\zeta'}\right) d_n\varphi$$

$$= \int_0^{2\pi} \cdots \int_0^{2\pi} F_q\left(\frac{z_1}{\tau_1 e^{i\psi_1} w}, \cdots, \frac{z_{n-1}}{\tau_{n-1} e^{i\psi_{n-1}} w}, \frac{z_n}{\tau_n w}\right) \ln|f(w; \kappa)| d_n\psi \tag{4.3.14}$$

$$= \frac{1}{\tau_1 \cdots \tau_n} \int_{P_\tau} \left\{ \int_0^{2\pi} \{\ln|\Pi(w; \kappa)| + \operatorname{Re} h(w; \kappa)\} F_q(w; z; \kappa)\, d\phi \right\} dW_\tau,$$

where $w = Re^{i\phi}$, and

$$F_q(w; z; \kappa) = F_q\left(\frac{z_1 e^{-i\psi_1}}{\tau_1 w}, \cdots, \frac{z_{n-1} e^{-i\psi_{n-1}}}{\tau_{n-1} w}, \frac{z_n}{\tau_n w}\right).$$

Since $h(w; \kappa)$ is an entire function of w, it admits the representation

$$h(w; \kappa) = \sum_{m=0}^{\infty} a_m(\kappa) w^m.$$

Consequently

$$\int_0^{2\pi} \operatorname{Re} h(w; \kappa) F_q(w; z; \kappa)\, d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \sum_{m=0}^{\infty} (a_m w^m + \bar{a}_m \bar{w}^m) \sum_{\|k\| > q} \frac{z^k \exp(-i(k_1\psi_1 + \cdots + k_{n-1}\psi_{n-1}))}{w^{\|k\|} \tau_1^{k_1} \cdots \tau_n^{k_n}} d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \sum_{m=0}^{\infty} (a_m w^m + \bar{a}_m \bar{w}^m) \cdot \sum_{l=q+1}^{\infty} A_l w^{-l} d\phi, \tag{4.3.15}$$

where

$$A_l = \sum_{\|k\| = l} \frac{z^k \exp(-i(k_1\psi_1 + \cdots + k_{n-1}\psi_{n-1}))}{\tau_1^{k_1} \cdots \tau_n^{k_n}}.$$

It now follows from (4.3.15) that

$$\int_0^{2\pi} \operatorname{Re} h(w; \kappa) F_q(w; z; \kappa)\, d\phi = \pi \sum_{l=q+1}^{\infty} a_l A_l.$$

Note that this sum is independent of R. Consequently the same is true of the integral

$$\int_{P_\tau} dW_\tau \int_0^{2\pi} \operatorname{Re} h(w; \kappa) F_q(w; z; \kappa)\, d\phi. \tag{4.3.16}$$

We now prove that

$$\lim_{R \to \infty} \int_{P_\tau} dW_\tau \int_0^{2\pi} \ln|\Pi(w; \kappa)| F_q(w; z; \kappa)\, d\phi = 0. \tag{4.3.17}$$

To this end we use the formula (see, for example, Nevanlinna [1])

$$\frac{1}{m!}\frac{d^m \ln f(w)}{dw^m}\bigg|_{w=0} = \frac{1}{\pi R^m}\int_0^{2\pi} \ln|f(Re^{i\psi})|e^{-im\psi}d\psi - \frac{1}{m}\sum_{|a_j|\leqq R}\frac{1}{a_j^m} + \frac{1}{m}\sum_{|a_j|\leqq R}\left(\frac{\bar{a}_j}{R^2}\right)^m. \tag{4.3.18}$$

Here $f(w)$ is a function analytic in the disk $|w| \leqq R$, nonvanishing for $w = 0$, and the numbers a_j are its zeros. Let us apply this formula to the function $\Pi(w; \kappa)$. For $m > q$ we get

$$\begin{aligned}\frac{1}{\pi R^m}\int_0^{2\pi} \ln|\Pi(Re^{i\psi}; \kappa)|\, e^{-im\psi}d\psi &= \frac{1}{m!}\frac{d^m \ln \Pi(w; \kappa)}{dw^m}\bigg|_{w=0} + \frac{1}{m}\sum_{|w_j|\leqq R} w_j^{-m} - \frac{1}{m}\sum_{|w_j|\leqq R}\left(\frac{\bar{w}_j}{R^2}\right)^m \\ &= -\frac{1}{m}\sum_{|w_j|>R} w_j^{-m} - \frac{1}{m}\sum_{|w_j|\leqq R}\left(\frac{\bar{w}_j}{R^2}\right)^m. \end{aligned}\tag{4.3.19}$$

This last equality follows from the fact that in every disk $|w| < R_0$ in which $\Pi(w; \kappa) \neq 0$ we have the obvious expansion

$$\ln \Pi(w; \kappa) = -\sum_{l=q+1}^{\infty} c_l w^l,$$

where

$$c_l = \frac{1}{l}\sum_{j=1}^{\infty} w_j^{-l}.$$

From (4.3.19) we immediately infer the estimate

$$\frac{1}{\pi R^m}\left|\int_0^{2\pi} \ln|\Pi(w; \kappa)|e^{-im\psi}\, d\psi\right| \leqq \sum_{|w_j|>R}|w_j|^{-m} + \frac{1}{m}\frac{n_f(R; \kappa)}{R^m},$$

and application of this estimate now shows that for $|z_i| < \tau_i R$, $i = 1, \cdots, n$,

$$\begin{aligned}&\left|\int_{P_\tau} dw_\tau \int_0^{2\pi} \ln|\Pi(w; \kappa)|\, F_q(w; z; \kappa)\, d\psi\right| \\ &= \left|\int_{P_\tau}\left\{\sum_{\|k\|>q}\frac{z^k \exp\left(-i(k_1\psi_1 + \cdots + k_{n-1}\psi_{n-1})\right)}{R^{\|k\|}\tau_1^{k_1}\cdots\tau_n^{k_n}}\int_0^{2\pi}\ln|\Pi(w; \kappa)|e^{-i\|k\|\psi}d\psi\right\}dW_\tau\right| \\ &\leqq \pi\sum_{\|k\|>q}\frac{|z_1|^{k_1}\cdots|z_n|^{k_n}}{\|k\|\tau_1^{k_1}\cdots\tau_n^{k_n}}\int_{P_\tau}\left(\frac{n_f(R; \kappa)}{R^{\|k\|}} + \int_R^{\infty}\frac{dn_f(t; \kappa)}{t^{\|k\|}}\right)dW_\tau \qquad (4.3.20) \\ &= \pi W_\tau\sum_{\|k\|>q}\frac{|z_1|^{k_1}\cdots|z_n|^{k_n}}{\|k\|\tau_1^{k_1}\cdots\tau_n^{k_n}}\left(\frac{n_{f,\tau}(R)}{R^{\|k\|}} + \int_R^{\infty}\frac{dn_{f,\tau}(t)}{t^{\|k\|}}\right) \\ &\leqq \pi W_\tau\sum_{\|k\|>q}\frac{|z_1|^{k_1}\cdots|z^n|^{k_n}}{\|k\|\tau^{k_1}\cdots\tau_n^{k_n}R^{\|k\|-q-1}}\left(\frac{n_{f,\tau}(R)}{R^{q+1}} + \int_R^{\infty}\frac{dn_{f,\tau}(t)}{t^{q+1}}\right).\end{aligned}$$

Since by assumption the integral

$$\int^{\infty} \frac{N_{f,\tau}(t)}{t^{q+1}}\, dt,$$

is convergent, which is equivalent to convergence of the integral

$$\int^{\infty} \frac{n_{f,\tau}(t)}{t^{q+2}}\, dt,$$

it follows that as $R \to \infty$

$$\int_R^{\infty} \frac{dn_{f,\tau}(t)}{t^{q+1}} \to 0, \qquad \frac{n_{f,\tau}(R)}{R^{q+2}} \to 0.$$

Together with (4.3.20), this implies (4.3.17), and moreover the convergence to zero in (4.3.17) is uniform in every bounded region of C^n. Consequently, by (4.3.14) and the fact that the integral (4.3.16) is independent of R, the following limit exists uniformly in every bounded region of C^n:

$$\lim_{R\to\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln|f(\zeta')|\, F_q\left(\frac{z}{\zeta'}\right) d_n\varphi = \frac{(2\pi)^n}{2} g(z). \tag{4.3.21}$$

Now the integral on the left of this relation is an analytic function in the polycylinder $\{z\colon |z_i| < R\tau_i,\ i = 1, \cdots, n\}$. Hence the function $g(z)$ defined by (4.3.21) is an entire function. This proves part (1) of our theorem.

To prove part (2), we again appeal to formula (4.3.18).[31] Set

$$Q_m(z) = \sum_{\|k\|=m} z^k, \qquad w = Re^{i\psi},$$
$$\lambda = (\tau_1 e^{i\psi_1}, \cdots, \tau_{n-1} e^{i\psi_{n-1}}, \tau_n).$$

Note that the rays κ defined by these vectors λ are in P_τ.

It is easy to see that

$$\begin{aligned}
\frac{1}{m!}\frac{\partial^m \ln f(wz)}{\partial w^m}\bigg|_{w=0} &= \frac{1}{m!W_\tau}\int_{P_\tau}\left(\frac{\partial^m}{\partial w^m}\ln(w\lambda)\right)_{w=0} Q_m\left(\frac{z}{\lambda}\right) dW_\tau(\lambda) \\
&= \frac{1}{\pi R^m W_\tau}\int_{P_\tau}\left\{\int_0^{2\pi}\ln|f(w\lambda)|e^{-im\psi}\, d\psi\right\} Q_m\left(\frac{z}{\lambda}\right) dW_\tau(\lambda) \\
&\quad - \frac{1}{mW_\tau}\int_{P_\tau}\left\{\sum_{|w_j|\le R} w_j^{-m}\right\} Q_m\left(\frac{z}{\lambda}\right) dW_\tau(\lambda) \\
&\quad + \frac{1}{mW_\tau}\int_{P_\tau}\left\{\sum_{|w_j|\le R}\left(\frac{\bar{w}_j}{R^2}\right)^m\right\} Q_m\left(\frac{z}{\lambda}\right) dW_\tau(\lambda) \\
&= \mathrm{I}_m + \mathrm{II}_m + \mathrm{III}_m.
\end{aligned}$$

[31] The idea of using formula (4.3.18) to derive representations of entire functions of several variables is due to Kneser [1].

Simple manipulations now give us

$$I_m = \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln|f(\zeta')| Q_m\left(\frac{z}{\zeta'}\right) d_n\varphi,$$

where $\zeta' = (R\tau_1 e^{i\varphi_1}, \cdots, R\tau_n e^{i\varphi_n})$;

$$\begin{aligned} II_m &= -\frac{1}{mW_\tau} \int_{P_\tau} \left\{ \sum_{|w_j| \leqq R} Q_m\left(\frac{z}{\lambda w_j}\right) \right\} dW_\tau(\lambda) \\ &= -\frac{1}{mW_\tau} \int_{\chi_{f,\tau}^R} \gamma_f(\zeta) Q_m\left(\frac{z}{\zeta}\right) dW_\tau(\zeta), \end{aligned}$$

where $\chi_{f,\tau}^R = \chi_{f,\tau} \cap \{\zeta : \zeta \in C^n, |\zeta| \leqq R\}$; and

$$|III_m| \leqq \frac{1}{mW_\tau R^m} \left\{ \sum_{j=1}^n \frac{|z_j|}{\tau_j} \right\}^m \int_{P_\tau} n_f(R; \kappa) dW_\tau = \frac{1}{mR^m} \left(\sum_{j=1}^n \frac{\tau_j}{|z_j|} \right)^m n_{f,\tau}(R).$$

Consequently for any $R > 0$

$$\begin{aligned} \sum_{w=q+1}^{\infty} \frac{1}{m!} \frac{\partial^m \ln f(wz)}{\partial w^m}\bigg|_{w=0} &= \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln|f(\zeta')| F_q\left(\frac{z}{\zeta'}\right) d_n\varphi \\ &\quad - \frac{1}{W_\tau} \int_{\chi_{f,\tau}^R} \gamma_f(\zeta) \Phi_q\left(\frac{z}{\zeta}\right) dW_\tau(\zeta) + \beta(z, R), \end{aligned} \tag{4.3.22}$$

where

$$\begin{aligned} |\beta(z, R)| &\leqq n_{f,\tau}(R) \sum_{m=q+1}^{\infty} \left(\frac{|z_1|\tau_1^{-1} + \cdots + |z_n|\tau_n^{-1}}{R} \right)^m \\ &= \frac{n_{f,\tau}(R)}{R^{q+1}} \left(\sum_{j=1}^n \frac{|z_j|}{\tau_j} \right)^{q+1} \left(1 - \sum_{j=1}^n \frac{|z_j|}{R\tau_j} \right)^{-1}. \end{aligned} \tag{4.3.22'}$$

We now let $R \to \infty$ in (4.3.22). As we mentioned above, the limit $g(z)$ of the first term on the right exists and is an entire function. Moreover, by (4.3.22'), $\beta(z, R) \to 0$ as $R \to \infty$, uniformly in z in any bounded region of C^n. Finally,

$$\lim_{R \to \infty} \int_{\chi_{f,\tau}^R} \gamma_f(\zeta) \Phi_q\left(\frac{z}{\zeta}\right) dW_\tau(\zeta) = \int_{\chi_{f,\tau}} \gamma_f(\zeta) \Phi_q\left(\frac{z}{\zeta}\right) dW_\tau(\zeta)$$

uniformly in $B = \{z : |z_i| < h\tau_i, i = 1, \cdots, n\}$, where h is a number such that $B \cap \chi_f = \varnothing$. Indeed, under these conditions,

$$M = \max_{\zeta \in \chi_{f,\tau}, z \in B} \left| \Phi_q\left(\frac{z}{\zeta}\right) \right| |\zeta|^{q+1} < \infty$$

and

$$\int_{\chi_{f,\tau}\setminus\chi_{f,\tau}^{R}} \gamma_f(\zeta)\left|\Phi_q\left(\frac{z}{\zeta}\right)\right| dW_\tau(\zeta) \leqq M \int_{\chi_{f,\tau}\setminus\chi_{f,\tau}^{R}} \gamma_f(\zeta)\frac{dW_\tau(\zeta)}{|\zeta|^{q+1}}$$

$$= M\int_{P_\tau}\left\{\sum_{|w_j|>R}|\omega_j|^{-q-1}\right\}dW_\tau$$

$$= M\int_{P_\tau} dW_\tau \int_R^\infty \frac{dn_f(t;\kappa)}{t^{q+1}} = MW_\tau\int_R^\infty \frac{dn_{f,\tau}(t)}{t^{q+1}},$$

and the last integral tends to zero as $R \to \infty$.

On the other hand, the left-hand side of (4.3.22) is independent of R and is the power-series expansion of $\ln f(z)$ in the domain B, with the terms of order lower than $q+1$ dropped. Thus in B

$$\ln f(z) = P_q(z) + g(z) - \frac{1}{W_\tau}\int_{\chi_{f,\tau}} \gamma_f(\zeta)\Phi_q\left(\frac{z}{\zeta}\right)dW_\tau(\zeta), \tag{4.3.23}$$

where $P_q(z)$ is a polynomial of degree at most q, and $g(z)$ the entire function defined by (4.3.21). This proves part (2) of our theorem.

To prove part (3), we first show that on every ray κ the function

$$\varphi(z) = \exp\left\{-\frac{1}{W_\tau}\int_{\chi_{f,\tau}} \gamma_f(\zeta)\Phi_q\left(\frac{z}{\zeta}\right)dW_\tau(\zeta)\right\},$$

which is obviously an entire function, is identical with the function $\Pi(w;\kappa)$.

Indeed, for $z = (\tau_1 e^{i\psi_1'}w, \cdots, \tau_{n-1}e^{i\psi_{n-1}'}w, \tau_n w) \in B$ we have

$$\ln\varphi(z) = -\frac{1}{W_\tau}\int_{\chi_{f,\tau}} \gamma_f(\zeta)\Phi_q\left(\frac{z}{\zeta}\right)dW_\tau(\zeta)$$

$$= -\frac{1}{W_\tau}\int_{P_\tau}\left\{\sum_j \Phi_q\left(\frac{z_1}{\tau_1 e^{i\psi_1}w_j}, \cdots, \frac{z_{n-1}}{\tau_{n-1}e^{i\psi_{n-1}}w_j}, \frac{z_n}{\tau_n w_j}\right)\right\}dW_\tau$$

$$= -\frac{1}{W_\tau}\int_{P_\tau}\left\{\sum_{\|k\|>q}\left\{\frac{w^{\|k\|}\, e^{i(k_1\psi_1'+\cdots+k_{n-1}\psi_{n-1}')}}{\|k\|\, e^{i(k_1\psi_1+\cdots+k_{n-1}\psi_{n-1})}}\sum_j w_j^{-\|k\|}\right\}\right\}dW_\tau. \tag{4.3.24}$$

As mentioned above, for $l > q$,

$$\frac{1}{l}\sum_{j=1}^{\infty} w_j^{-l} = -c_l(\kappa), \tag{4.3.25}$$

where $c_l(\kappa)$ is defined as the corresponding coefficient in the expansion

$$\ln\Pi(w;\kappa) = \sum_{l=q+1}^{\infty} c_l(\kappa)w^l. \tag{4.3.26}$$

It now follows from Theorem 4.3.3 and the definition of the ray κ that $c_l(\kappa)$ is a trigonometric polynomial of degree l in $\psi_1, \cdots, \psi_{n-1}$:

$$c_l(\kappa) = c_l(\psi_1, \cdots, \psi_{n-1}) = \sum_{\|k\|=l} a_k\, e^{i(k_1\psi_1+\cdots+k_{n-1}\psi_{n-1})}. \tag{4.3.27}$$

From (4.3.24)—(4.3.27) we conclude that for $|w| < h$, where h is the number figuring in the definition of the domain B,

$$\ln \varphi(\tau_1 e^{i\psi_1'} w, \cdots, \tau_{n-1} e^{i\psi_{n-1}'} w, \tau_n w) = -\sum_{\|k\|>q} a_k w^{\|k\|} e^{i(k_1\psi_1' + \cdots + k_{n-1}\psi_{n-1}')}$$

$$= \sum_{l=q+1}^{\infty} w^l c_l(\psi_1', \cdots, \psi_{n-1}') = \ln \Pi(w; \kappa')$$

where the ray κ' is defined by the point $(\tau_1 e^{i\psi_1'}, \cdots, \tau_{n-1} e^{i\psi_{n-1}'}, \tau_n)$. It immediately follows that the function $\varphi(z)$ coincides with $\Pi(w; \kappa')$ on the whole ray κ', $\kappa' \in P_\tau$, i.e. for $|w| < \infty$.[32]

We shall use this property of the function $\varphi(z)$ to estimate its growth. By the above-mentioned estimate of the canonical product,

$$\ln^+ |\varphi(\tau_1 e^{i\psi_1} w, \cdots, \tau_{n-1} e^{i\psi_{n-1}} w, \tau_n w)| \leqq C_{q,n} K_q(R; n_f(t; \kappa)) \qquad (w = Re^{i\psi}).$$

Consequently

$$\frac{1}{2\pi} \int_0^{2\pi} \ln^+ |\varphi(\tau_1 e^{i\psi_1} w, \cdots, \tau_{n-1} e^{i\psi_{n-1}} w, \tau_n w)| d\psi \leqq C_{n,q} K_q(R; n_f(t; \kappa))$$

and

$$\frac{1}{2\pi W_\tau} \int_{P_\tau} dW_\tau \int_0^{2\pi} \ln^+ |\varphi(\tau_1 e^{i\psi_1} w, \cdots, \tau_{n-1} e^{i\psi_{n-1}} w, \tau_n w)| d\psi \leqq \frac{C_{n,q}}{W_\tau} \int_{P_\tau} K_q(R; n_f(t; \kappa)) dW_\tau.$$

Changing the order of integration in the right-hand side and performing the substitution $\varphi_i = \psi_i + \psi$, $i < n$, $\varphi_n = \psi$, in the left-hand side, we get

$$\left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \ln^+ |\varphi(\tau_1 e^{i\varphi_1} R, \cdots, \tau_n e^{i\varphi_n} R)| d_n\varphi \leqq C_{n,q} K_q(R; n_{f,\tau}(t)). \tag{4.3.28}$$

The integral on the left of this inequality is the average of the plurisubharmonic function $\ln^+ |\varphi(z)|$ over the distinguished boundary of the polycylinder $\{z: |z_i| \leqq r_i = \tau_i R, i = 1, \cdots, n\}$. It is therefore a monotone nondecreasing function of $r_1, \cdots, r_n$. Consequently

$$\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln^+ |\varphi(\tau_1 e^{i\varphi_1} R, \cdots, \tau_n e^{i\varphi_n} R)| d_n\varphi$$

$$\geqq \frac{(i/2)^n}{\pi^n \tau_1^2 \cdots \tau_n^2 R^{2n}} \int_{\substack{|z_i| < R\tau_i \\ i=1,\cdots,n}} \ln^+ |\varphi(z)| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n$$

$$\geqq \frac{(i/2)^n}{\pi^n \tau_1^2 \cdots \tau_n^2 R^{2n}} \int_D \ln^+ |\varphi(z)| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n, \tag{4.3.29}$$

where $D = \{z: |z_i - z_i'| < \varepsilon \tau_i R, i = 1, \cdots, n\}$, ε is any number from the interval (0, 1), and the only condition imposed on z_i' is

[32] This of course implies that on any ray $\kappa \in P_\tau$ the function $g(z) + P_q(z)$ coincides with $h(w; \kappa)$.

$$|z_i'| = (1 - \varepsilon)\tau_i R.$$

By the mean value theorem for plurisubharmonic functions,

$$\ln^+|\varphi(z')| \leqq \frac{(i/2)^n}{\varepsilon^n \pi^n R^{2n} \tau_1^2 \cdots \tau_n^2} \int_D \ln^+|\varphi(z)| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

Together with (4.3.29), this implies that

$$\left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \ln^+|\varphi(\tau_1 e^{i\varphi_1} R, \cdots, \tau_n e^{i\varphi_n} R)| d_n\varphi \geqq \varepsilon^n \ln^+|\varphi(z')|.$$

By inequality (4.3.28), we now conclude that for any $R > 0$

$$\ln^+ M_\varphi(\tau_1(1 - \varepsilon)R, \cdots, \tau_n(1 - \varepsilon)R) \leqq \varepsilon^{-n} C_{n,q} K_q(R; n_{f,\tau}(t)). \tag{4.3.30}$$

It follows from this estimate that the order of the function $\varphi(z)$ is equal to the order ρ of $n_{f,\tau}(t)$, and when $\rho = q + 1$ the function $\varphi(z)$ is of minimal type. That the equality $\varphi = \varphi^* e^{-P_*}$ holds is obvious. In addition, by (4.3.23) we have $f = \varphi e^{g+P_*}$. Consequently $\varphi(z)$ is the canonical function, and the proof is complete.

4. *Entire functions of minimal growth in one of the variables.* As we mentioned above, the order $\rho_n(r_1, \cdots, r_{n-1}; f)$ of the function $\ln M_f(r_1, \cdots, r_n)$ in the variable r_n is independent of the fixed values of the other variables. The same holds for the order $\rho_n(r_1, \cdots, r_{n-1}; N_f)$ of the function $N_{f,z_n}(t; r_1, \cdots, r_{n-1})$ in the variable t. This makes it reasonable to expect that there exists an entire function $\varphi_n(z)$ with the same zeros as $f(z)$, which in some sense is of minimal growth with respect to the variable z_n; in particular, such that $\bar{\rho}_n(N_f) = \bar{\rho}_n(\varphi_n)$. Here we shall prove the existence of such functions $\varphi_n(z)$ under certain restrictions on the function $N_f(r)$. We shall also give an integral representation of the function $\ln \varphi_n(z)$ as an integral over the manifold

$$\chi_{f,r^*} = \chi_f \cap \left\{z: \underset{i=1,\cdots,n-1}{|z_i| = r_i}, |z_n| < \infty\right\}.$$

To simplify the notation, we supplement our standard notation with the following:

$$\mathscr{E}(r^*) = \left\{z^*: \underset{i=1,\cdots,n-1}{|z_i| \leqq r_i}\right\}, \qquad E_f = \{z^*: f(z^*, 0) \neq 0\},$$

$$\frac{\zeta^*}{\zeta^* - z^*} = \prod_{i=1}^{n-1} \frac{\zeta_i}{\zeta_i - z_i},$$

where, as in § 2, the star indicates omission of the last coordinate.

As in § 3.3, our proof of the existence of the required functions is based on a theorem on the zeros of entire functions:

THEOREM 4.3.5 (RONKIN [12]). *Let $f(z)$ be an entire function such that $\rho(N_f) < \infty$, $f(0) \neq 0$. Let $z_{n,j}(z^*)$, $j = 1,2, \cdots$, be the zeros of the function $f(z^*, z_n)$, indexed in order of increasing absolute value. Then for any $m > \bar{\rho}_n(N_f)$ the function*

$$\frac{1}{m!}\frac{\partial^m \ln f}{\partial z_n^m}\bigg|_{z_n=0} + \frac{1}{m}\sum_{j=1}^{\infty} z_{n,j}^{-m}(z^*),$$

defined in the domain E_f, can be continued to the entire space C^{n-1} as an entire function.

PROOF. Consider the function $f_l(z) = f(z^*, z_n^l)$. It is obvious that

$$N_{f_l}(r_1, \cdots, r_n) = N_f(r_1, \cdots, r_{n-1}, r_n^l)$$

and hence $\rho(N_{f_l}) < \infty$. Hence by Theorem 4.3.4 the function $f_l(z)$ admits the representation

$$f_l(z) = e^{g_l(z)}\psi_l(z), \tag{4.3.31}$$

where $g(z)$ is an entire function, and the order ρ_{ψ_l} of the function $\psi_l(z)$ is the same as that of the function $N_{f_l}(r)$, or, equivalently, the same as that of $N_f(r_1,\cdots, r_{n-1}, r_n^l)$. It is clear that the order of the latter function is equal to the value of the parameter t defining the point of intersection of the hypersurface of associated orders of $N_f(r_1, \cdots, r_n)$ with the straight line $a_1 = \cdots = a_{n-1} = t$, $a_n = t/l$. Hence by Theorem 2.6.4 we conclude that for any $\varepsilon > 0$ and sufficiently large l

$$l^{-1}\rho_{\psi_l} - \bar{\rho}_n(N_f) < \varepsilon. \tag{4.3.32}$$

We now apply (4.3.18) to the function $f(z)$. For any $R > 0$ and $z^* \in E_f$ we have

$$\frac{1}{m!}\frac{\partial^m \ln f}{\partial z_n^m}\bigg|_{z_n=0} + \frac{1}{m}\sum_{|z_{n,j}|\le R} z_{n,j}^{-m}(z^*)$$
$$= \frac{1}{\pi R^m}\int_0^{2\pi} \ln\left|f(z^*, Re^{i\varphi_n})\right| e^{-im\varphi_n} d\varphi_n + \frac{1}{m}\sum_{|z_{n,j}|\le R}\left(\frac{\bar{z}_{n,j}}{R^2}\right)^m.$$

Setting $R = r_n^l$ and $\varphi_n = l\theta$, we get

$$\frac{1}{\pi R^m}\int_0^{2\pi} \ln\left|f(z^*, Re^{i\varphi_n})\right| e^{-im\varphi_n} d\varphi_n + \frac{1}{m}\sum_{|z_{n,j}|\le R}\left(\frac{\bar{z}_{n,j}}{R^2}\right)^m$$
$$= \frac{1}{\pi r_n^{lm}}\int_0^{2\pi} \ln\left|f_l(z^*, r_n e^{i\theta})\right| e^{-iml\theta} d\theta + \frac{1}{m}\sum_{|z_{n,j}|\le R}\left(\frac{\bar{z}_{n,j}}{R^2}\right)^m.$$

Together with (4.3.31), this implies that

$$\frac{1}{m!}\frac{\partial^m \ln f}{\partial z_n^m}\bigg|_{z_n=0} + \frac{1}{m}\sum_{|z_{n,j}|\le R} z_{n,j}^{-m}$$
$$= \frac{1}{\pi r_n^{ml}}\int_0^{2\pi} \operatorname{Re} g_l(z^*, r_n e^{i\theta}) e^{-iml\theta} d\theta + \frac{1}{\pi r_n^{ml}}\int_0^{2\pi} \ln\left|\psi_l(z^*, r_n e^{i\theta})\right| e^{-iml\theta} d\theta$$
$$+ \sum_{|z_{n,j}|\le R}\left(\frac{\bar{z}_{n,j}}{R^2}\right)^m = \mathrm{I}_m + \mathrm{II}_m + \mathrm{III}_m.$$

Now I_m is an entire function, independent of r_n. Indeed, the function $g_l(z)$ is entire, and can therefore be expressed in the form

$$g_l(z) = \sum_{p=0}^{\infty} a_p^{(l)}(z^*) z_n^p,$$

where the coefficients $a_p^{(l)}(z^*)$ are entire functions. Therefore

$$\mathrm{I}_m = \frac{1}{2\pi r_n^{ml}} \int_0^{2\pi} \left\{ \sum_{p=0}^{\infty} a_p^{(l)}(z^*) r_n^p e^{ip\theta} + \sum_{p=0}^{\infty} \bar{a}_p^{(l)}(z^*) r_n^p e^{-ip\theta} \right\} e^{-im\theta} d\theta = a_{ml}^{(l)}(z^*),$$

whence it follows that I_m is an entire function independent of r_n.

Now, using inequality (4.3.32) and Jensen's formula, we estimate II_m for $r_n \to \infty$. We have

$$\begin{aligned}
|\mathrm{II}_m| &\leqq \frac{1}{\pi r_n^{ml}} \int_0^{2\pi} |\ln|\psi_l(z^*, r_n e^{i\theta})|| d\theta = \frac{1}{\pi r_n^{ml}} \int_0^{2\pi} (\ln^+|\psi_l| + \ln^-|\psi_l|) d\theta \\
&= \frac{1}{\pi r_n^{ml}} \left\{ 2 \int_0^{2\pi} \ln^+|\psi_l| d\theta - N_{\psi_l}(z^*, r_n) + \alpha(z^*) \right\} \\
&\leqq \frac{4 \ln^+ M_{\psi_l}(|z_1|, \cdots, |z_{n-1}|, r_n) + \alpha(z^*)}{r_n^{ml}} \\
&\leqq 4(C_\varepsilon(z^*) + r_n^{\rho_{\psi_l} + l\varepsilon}) r_n^{-ml} \leqq 4(C_\varepsilon(z^*) + R^{\bar{\rho}_n(N_f) + 2\varepsilon}) R^{-m}.
\end{aligned}$$

Here $\varepsilon > 0$ is arbitrary, and the function $C_\varepsilon(z^*)$ is bounded in every compact subset of E_f. It follows from this estimate that for all $m > \bar{\rho}_n(N_f)$ we have $\mathrm{II}_m \to 0$ as $r_n \to \infty$, and this holds uniformly on every compact subset of E_f.

We now estimate III_m. We have

$$|\mathrm{III}_m| \leqq \sum_{|z_{n,j}| \leqq R} \frac{|z_{n,j}|^m}{R^{2m}} \leqq \frac{n_{f,z_n}(R; z^*)}{R^m}.$$

As we mentioned above, the order of the function $n_{f,z_n}(t; r^*)$ in the variable t is equal to that of the function $N_f(r)$ in the variable r_n. Hence, using Theorem 4.2.8, we conclude that for all $m > \bar{\rho}_n(N_f)$, $\mathrm{III}_m \to 0$ as $R \to \infty$, and this holds uniformly on every bounded region in C^{n-1}.

Combining the above properties of the functions I_m, II_m and III_m, we see that for $m > \bar{\rho}_n(N_f)$ and $z^* \in E_f$

$$\frac{1}{m!} \frac{\partial^m \ln f}{\partial z_n^m}\bigg|_{z_n=0} + \frac{1}{m} \sum_{j=0}^{\infty} z_{n,j}^{-m}(z^*) = a_{ml}^{(l)}(z^*),$$

and $a_{ml}^{(l)}(z^*)$, as remarked, is an entire function.

This completes the proof.

As a corollary of Theorem 4.3.5, it follows that *for* $m > \bar{\rho}_n(N_f)$ *the sum* $\sum_{j=1}^{\infty} z_{n,j}^{-m}(z^*)$ *in Theorem* 4.3.5 *is analytic in* E_f *and meromorphically continuable to the whole space* C^{n-1}.

THEOREM 4.3.6 (RONKIN [12]). *Let $f(z)$ be an entire function such that $\rho(N_f) < \infty$ and $\bar{\rho}_n(N_f) \neq q + 1$, where q is the smallest integer for which the integral*

$$\int^{\infty} \frac{N_f(r^*, r_n)}{r_n^{q+2}}\, dr_n \tag{4.3.33}$$

is convergent. Let $f(z^*, z_n) \not\equiv 0$ for every fixed z^*, and $f(0) \neq 0$. Then the function $f(z)$ admits the representation

$$f(z) = e^{g_n(z)}\varphi_n(z),$$

where $g_n(z)$ is an entire function and the function $\varphi_n(z)$ is such that $\bar{\rho}_n(\varphi_n) = \bar{\rho}_n(N_f)$. This representation has the following properties:

(1) *The function $g_n(z)$ admits the following representation in every polycylinder $\{z: z^* \in \mathscr{E}(r^*), |z_n| < \infty\}$:*

$$g_n(z) = \lim_{r_n \to \infty} \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} F_q^{(n)}\left(\frac{z}{\zeta}\right) \ln|f(\zeta)|\, d_n\psi, \tag{4.3.34}$$

where

$$F_q^{(n)}(z) = \frac{z_n^{q+1}}{(1 - z_1)\cdots(1 - z_n)}, \qquad \zeta = (r_1 e^{i\psi_1}, \cdots, r_n e^{i\psi_n}).$$

(2) *The function $\varphi_n(z)$ admits the following representation in every polycylinder $\{z: z^* \in \mathscr{E}(r^*), |z_n| < \infty\}$ such that $\mathscr{E}(r^*) \subset E_f$:*

$$\varphi_n(z) = \Psi_n(z) \exp\left\{ \sum_{m=0}^{q} \frac{z_n^m}{m!} \left.\frac{\partial^m \ln f}{\partial z_n^m}\right|_{z_n=0} \right\}, \tag{4.3.35}$$

where the function $\Psi_n(z)$ is holomorphic in the polycylinder $\{z: z^* \in \mathscr{E}(r^*), |z_n| < \infty\}$ and is defined in the polydisk $\{z: z^* \in \mathscr{E}(r^*), |z_n| < r_n\}$ for sufficiently small $r_n > 0$ by the equality[33]

$$\ln \Psi_n(z) = -\frac{1}{(2\pi)^{n-1}} \int_{\chi_{f,r^*}} \gamma_f(\zeta) \Phi_q^{(n)}\left(\frac{z}{\zeta}\right) d_{n-1}\psi,$$

where

$$\Phi_q^{(n)}(z) = \prod_{i=1}^{n-1} \frac{1}{1 - z_i} \sum_{l=q+1}^{\infty} \frac{z_n^l}{l}.$$

(3) *For any $\delta > 1$ and $\delta_1 > 1$ there exists a constant $A = A(\delta, \delta_1, r^*)$ such that for all r_n*

$$\ln M_{\Psi_n}(r^*, r_n) \leqq A K_q(r_n; N_{f,z_n}(\delta_1 t; \delta r^*)).$$

The proof of this theorem is based on the same arguments as that of Theorem

[33] Recall that $\chi_{f,r^*} = \chi_j \cap \{z: |z_i| = r_i,\ i = 1, \cdots, n-1;\ |z_n| < \infty\}$.

4.3.4. We first show that the function defined by (4.3.34) is an entire function.

By Theorem 4.2.8 and equality (4.2.43) it follows from the convergence of the integral (4.3.33) that for any z^*

$$\int^{\infty} \frac{N_{f,z_n}(t; z^*)}{t^{q+2}}\, dt < \infty.$$

Hence the function $f(z) = f(z^*, z_n)$ admits the representation

$$f(z^*, z_n) = e^{h(z_n; z^*)} \Pi(z_n; z^*) \exp\left\{ \sum_{m=0}^{q} \frac{z_n^m}{m!} \left. \frac{\partial^m \ln f}{\partial z_n^m} \right|_{z_n=0} \right\}, \tag{4.3.36}$$

where $h(z_n; z^*)$ is an entire function of z_n, and

$$\Pi(z_n; z^*) = z_n^{\mu} \prod_{j=1}^{\infty} \left(1 - \frac{z_n}{z_{n,j}}\right) \exp\left\{ \sum_{m=1}^{q} \frac{1}{m} \left(\frac{z_n}{z_{n,j}}\right)^m \right\}.$$

Here $z_{n,j} = z_{n,j}(z^*)$ denote the nonzero roots of the function $f(z^*, z_n)$, regarded as a function of one variable z_n, and $\mu = \mu(z^*)$ is the multiplicity of the root of this function at $z_n = 0$. Substitute this expression into (4.3.34). We get

$$\frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} F_q^{(n)}\left(\frac{z}{\zeta}\right) \ln|f(\zeta)|\, d_n\psi = \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} F_q^{(n)}\left(\frac{z}{\zeta}\right) \operatorname{Re} h(\zeta_n; \zeta^*)\, d_n\psi$$

$$+ \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} F_q^{(n)}\left(\frac{z}{\zeta}\right) \ln|\Pi(\zeta_n; \zeta^*)|\, d_n\psi = A_1 + A_2.$$

As an entire function of z_n, the function $h(z_n; z^*)$ has the expansion

$$h(z_n; z^*) = \sum_{l=q+1}^{\infty} a_l(z^*) z_n^l.$$

It is readily seen that for $z^* \in E_f$

$$a_l(z^*) = \frac{1}{l!} \left. \frac{\partial^l \ln f}{\partial z_n^l} \right|_{z_n=0} + \frac{1}{l} \sum_{j=1}^{\infty} z_{n,j}^{-1}.$$

Consequently by Theorem 4.3.5 the coefficients $a_l(z^*)$ are entire functions. The details of the proof of Theorem 4.3.5 also show that the series $\sum_{l=q+1}^{\infty} a_l(z^*) z_n^l$ converges uniformly in every bounded region of C^{n-1}, and consequently $h(z_n; z^*)$ is an entire function in C^n. Taking this into account, we get

$$A_1 = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\zeta^*}{\zeta^* - z^*} \left\{ \int_0^{2\pi} \sum_{l=q+1}^{\infty} (a_l(\zeta^*)\zeta_n^l + a_l(\zeta^*)\bar{\zeta}_n^l) \sum_{l=q+1}^{\infty} \frac{z_n^l}{\zeta_n^l}\, d\psi_n \right\} d_{n-1}\psi$$

$$= \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \left\{ \frac{\zeta^*}{\zeta^* - z^*} \sum_{l=q+1}^{\infty} a_l(\zeta^*)\, z_n^l \right\} d_{n-1}\psi = \sum_{l=q+1}^{\infty} a_l(z^*) z_n^l = h(z_n; z^*).$$

We now apply (4.3.18) to the function $\Pi(z_n; z^*)$:

$$\frac{1}{\pi r_n^m}\int_0^{2\pi} e^{-im\phi_n}\ln|\Pi(r_n e^{i\phi_n};\zeta^*)|\,d\phi_n = \frac{1}{m!}\left(\frac{\partial^m}{\partial\zeta_n^m}\ln\Pi(\zeta_n;\zeta^*)\right)_{\zeta_n=0} + \frac{1}{m}\sum_{|z_{n,j}|\leqq r_n} z_{n,j}^{-m} - \frac{1}{m}\sum_{|z_{n,j}|\leqq r_n}\left(\frac{\bar z_{n,j}}{r_n^2}\right)^m$$

$$= -\frac{1}{m}\sum_{|z_{n,j}|> r_n} z_{n,j}^{-m} - \frac{1}{m}\sum_{|z_{n,j}|\leqq r_n}\left(\frac{\bar z_{n,j}}{r_n^2}\right)^m.$$

This implies the estimate

$$\frac{1}{\pi r_n^m}\left|\int_0^{2\pi}\dot{\ln}|\Pi(\zeta_n;\zeta^*)|e^{-im\phi_n}d\phi_n\right| \leqq \frac{1}{m}\sum_{|z_{n,j}|>r_n}|z_{n,j}|^{-m} + \frac{n_{f,z_n}(r_n;\zeta^*)}{mr_n^m}$$

$$= \frac{1}{m}\int_{r_n}^{\infty}\frac{dn_{f,z_n}(t;\zeta^*)}{t^m} + \frac{n_{f,z_n}(r;\zeta^*)}{mr_n^m}.$$

Further, we estimate A_2 for $|z_i| < r_i$, $i = 1, \cdots, n$. We have

$$|A_2| \leqq \frac{1}{\pi}\left|\int_0^{2\pi}\cdots\int_0^{2\pi}\left\{\int_0^{2\pi}\ln|\Pi(\zeta_n;\zeta^*)|\sum_{m=q+1}^{\infty}\left(\frac{z_n}{\zeta_n}\right)^m d\phi_n\right\}\frac{\zeta^*}{\zeta^*-z^*}\,d_{n-1}\psi\right|$$

$$\leqq \frac{1}{\pi}\left|\int_0^{2\pi}\cdots\int_0^{2\pi}\left\{\frac{\zeta^*}{\zeta^*-z^*}\sum_{m=q+1}^{\infty}\frac{z_n^m}{r_n^m}\int_0^{2\pi}\ln|\Pi(r_n e^{i\phi_n};\zeta^*)|e^{-im\phi_n}d\phi_n\right\}d_{n-1}\psi\right|$$

$$\leqq \prod_{i=1}^{n-1}\frac{r_i}{r_i-|z_i|}\sum_{m=q+1}^{\infty}\frac{|z_n|^m}{m}\left(\frac{n_{f,z_n}(r_n;r^*)}{r_n^m} + \int_{r_n}^{\infty}\frac{dn_{f,z_n}(t;r^*)}{t^m}\right)$$

$$\leqq \prod_{i=1}^{n-1}\frac{r_i}{r_i-|z_i|}\left(\frac{n_{f,z_n}(r_n;r^*)}{r_n^{q+1}} + \int_{r_n}^{\infty}\frac{dn_{f,z_n}(t;r^*)}{t^{q+1}}\right)\sum_{m=q+1}^{\infty}\frac{|z_n|^m}{mr_n^{m-q-1}}.$$

Note that the series

$$\sum_{m=q+1}^{\infty}\frac{|z_n|^m}{mr_n^{m-q-1}}$$

is convergent, provided $|z_n| < r_n$. Moreover, by the convergence of the integral (4.3.33) and the relation established in § 2 between $N_f(r)$ and $n_{f,z_n}(t;r^*)$ we have

$$\int_{r_n}^{\infty}\frac{dn_{f,z_n}(t;r^*)}{t^{q+1}} \to 0, \qquad \frac{n_{f,z_n}(r_n;r^*)}{r_n^{q+1}} \to 0$$

as $r_n \to \infty$. We thus see that A_1 is independent of $r_1, \cdots, r_n$ and is an entire function, while $A_2 \to 0$ as $r_n \to \infty$; moreover, as is easily seen, this convergence is uniform in $z^* \in \{z^*: |z_i| < (1-\varepsilon)r_i, i = 1, \cdots, n-1\}$, where $\varepsilon \in (0, 1)$ is arbitrary.

Consequently the limit

$$\lim_{r_n\to\infty}(A_1 + A_2) = g_n(z)$$

exists, $g_n(z)$ is an entire function independent of the choice of the numbers $r_1, \cdots, r_{n-1}$, and for any z^* we have

$$g_n(z) = h(z_n; z^*). \tag{4.3.37}$$

To prove that $\varphi_n(z) = f(z)e^{-g_n(z)}$ can be represented in the form (4.3.35), we apply (4.3.18) to $f(z)$:

$$\frac{1}{m!}\left(\frac{\partial^m}{\partial \zeta_n^m} \ln f(\zeta^*, \zeta_n)\right)_{\zeta_n=0} = \frac{1}{\pi r_n^m} \int_0^{2\pi} \ln\left|f(\zeta^*, r_n e^{i\phi_n})\right| e^{-im\phi_n} d\phi_n$$
$$- \frac{1}{m} \sum_{|z_{n,j}| \leqq r_n} z_{n,j}^{-m} + \frac{1}{m} \sum_{|z_{n,j}| \leqq r_n} \left(\frac{\bar{z}_{n,j}}{r_n^2}\right)^m.$$

The left-hand side of this equality is obviously an analytic function in any polydisk $\mathscr{E}(r^*) \subset E_f$. Therefore, by Cauchy's formula,

$$\begin{aligned}
&\frac{1}{m}\left(\frac{\partial^m}{\partial \zeta_n^m} \ln f(z^*, \zeta_n)\right)_{\zeta_n=0} \\
&= \frac{1}{(2\pi)^{n-1}} \cdot \frac{1}{\pi r_n^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\zeta^*}{\zeta^* - z^*} \ln\left|f(\zeta^*, r_n e^{i\phi_n})\right| e^{-im\phi_n} d_n\phi \\
&\quad - \frac{1}{m(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\zeta^*}{\zeta^* - z^*} \sum_{|z_{n,j}| \leqq r_n} z_{n,j}^{-m}(\zeta^*)\, d_{n-1}\psi \\
&\quad + \frac{1}{m(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\zeta^*}{\zeta^* - z^*} \sum_{|z_{n,j}| \leqq r_n} \left(\frac{\bar{z}_{n,j}}{r_n^2}\right)^m d_{n-1}\psi \\
&= \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\ln\left|f(\zeta)\right|}{\zeta_n^m} \frac{\zeta^*}{\zeta^* - z^*}\, d_n\psi \\
&\quad - \frac{1}{m(2\pi)^{n-1}} \int_{\chi_{f,r^*}^{r_n}} \zeta_n^{-m} \frac{\zeta^*}{\zeta^* - z^*}\, d_{n-1}\psi + \beta_m(r_n; z^*),
\end{aligned} \tag{4.3.38}$$

where

$$\chi_{f,r^*}^{r_n} = \chi_{f,r^*} \cap \{z: |z_n| \leqq r_n\},$$

and, as is readily seen, for any $z^* \in \mathscr{E}(r^*)$,

$$\left|\beta_m(r; z^*)\right| \leqq \frac{1}{m(2\pi)^{n-1}} \frac{n_{f,z_n}(r_n; r^*)}{r_n^m} \prod_{i=1}^{n-1} \frac{r_i}{r_i - |z_i|}.$$

Multiplying both sides of (4.3.38) by z_n^m and summing over $m \geqq q+1$, we get

$$\sum_{m=q+1} \frac{z_n^m}{m!} \left.\frac{\partial^m \ln f}{\partial z_n^m}\right|_{z_n=0} = \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln\left|f(\zeta)\right| F_q^{(n)}\left(\frac{z}{\zeta}\right) d_n\psi$$
$$- \frac{1}{(2\pi)^{n-1}} \int_{\chi_{f,r^*}^{r_n}} \gamma_f(\zeta) \Phi_q^{(n)}\left(\frac{z}{\zeta}\right) d_{n-1}\psi + \beta(r_n; z), \tag{4.3.39}$$

where for $z^* \in \mathscr{E}(r^*)$

$$|\beta(r_n; z)| \leqq \frac{1}{(2\pi)^{n-1}} \frac{n_{f,z_n}(r_n; r^*)}{r_n^{q+1}} \prod_{i=1}^{n-1} \frac{r_i}{r_i - |z_i|} \cdot \sum_{m=q+1}^{\infty} \frac{|z_n|^m}{m r_n^{m-q-1}}$$

and so $\beta(r_n; z) \to 0$ for $r_n \to \infty$, uniformly in z^* in any polydisk which lies entirely within $\mathscr{E}(r^*)$. Moreover, for $z^* \in \mathscr{E}(r^*)$ the integral

$$\int_{\chi_{f,r^*}} \gamma_f(\zeta) \Phi_q^{(n)}\left(\frac{z}{\zeta}\right) d_{n-1}\psi$$

exists, and is even absolutely convergent. Indeed,

$$\begin{aligned}
\int_{\chi_{f,r^*}} \gamma_1(\zeta) \left| \Phi_q^{(n)}\left(\frac{z}{\zeta}\right) \right| d_{n-1}\psi &\leqq \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{j=1}^{\infty} \left| \frac{\zeta^*}{\zeta^* - z^*} \sum_{m=q+1}^{\infty} \frac{1}{m} \left(\frac{z_n}{z_{n,j}(\zeta^*)}\right)^m \right| d_{n-1}\psi \\
&\leqq \prod_{i=1}^{n-1} \frac{r_i}{r_i - |z_i|} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{m=q+1}^{\infty} \left\{ |z_n|^m \sum_{j=1}^{\infty} |z_{n,j}|^{-m} \right\} d_{n-1}\psi \\
&\leqq (2\pi)^{n-1} \prod_{i=1}^{n-1} \frac{r_i}{r_i - |z_i|} \sum_{m=q+1}^{\infty} \int_0^{\infty} \frac{|z_n|^m \, dn_{f,z_n}(t; r^*)}{t^m} \\
&= (2\pi)^{n-1} |z_n|^{q+1} \prod_{i=1}^{n-1} \frac{r_i}{r_i - |z_i|} \int_0^{\infty} \frac{dn_{f,z_n}(t; r^*)}{t^q (t - |z_n|)}.
\end{aligned}$$

Consequently, letting $r_n \to \infty$ in (4.3.39), we get the relation

$$\begin{aligned}
\sum_{m=q+1}^{\infty} \frac{z_n^m}{m!} \cdot \left. \frac{\partial^m \ln f}{\partial z_n^m} \right|_{z_n=0} &= g_n(z) - \frac{1}{(2\pi)^{n-1}} \int_{\chi_{f,r^*}} \gamma_f(\zeta) \Phi_q^{(n)}\left(\frac{z}{\zeta}\right) d_{n-1}\psi \\
&= g_n(z) + \ln \Psi_n(z),
\end{aligned}$$

whence

$$f(z) = e^{g_n(z)} \Psi_n(z) \exp\left\{ \sum_{m=0}^{q} \frac{z_n^m}{m!} \cdot \left. \frac{\partial^m \ln f}{\partial z_n^m} \right|_{z_n=0} \right\}. \tag{4.3.40}$$

We now estimate the function $\Psi_n(z)$ for $z^* \in \mathscr{E}(r^*) \subset E_f$. It follows from (4.3.36) and (4.3.37) that $\Psi_n(z^*, z_n) = \Pi(z_n; z^*)$. Consequently, by the known estimate for the canonical product,

$$\ln^+ M_{\Psi_n}(z^*, r_n) \leqq C_{n,q} K_q(r_n; n_{f,z_n}(t; z^*)),$$

and therefore

$$\frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln^+ M_{\Psi_n}(z^*, r_n) \, d_{n-1}\psi \leqq C_{n,q} K_q(r_n; n_{f,z_n}(t; r^*)),$$

where $z_j = r_j e^{i\psi_j}$, $j = 1, \cdots, n-1$. Since the function

$$\ln^+ M_{\Psi_n}(z_n^*, r_n) = \max_{|z_n| = z_n} \ln^+ |\Psi_n(z^*, z_n)|$$

is plurisubharmonic in $z_1, \cdots, z_{n-1}$, we can proceed as in the proof of Theorem 4.3.4 and derive the following estimate, which holds for $z^* \in \mathscr{E}(r^*) \subset E_f$, any z_n and any $\delta > 0$:

$$\ln^+ M_{\Psi_n}(r^*, r_n) \leqq C_{\delta,n,q} K_q(r_n; n_{f,z_n}(t; \delta r^*)).$$

Here $C_{\delta,n,q}$ is a constant. Using the relation between the functions $n_{f,z_n}(t; r^*)$ and $N_{f,z_n}(t; r^*)$, we see further that for any $\delta_1 > 0$

$$\ln^+ M_{\Psi_n}(r^*, r_n) \leqq \frac{C_{\delta,q,n}}{\ln \delta_1} K_q(\delta_1 r_n; N_{f,z_n}(t; \delta r^*)).$$

Consequently $\bar{\rho}_n(\Psi_n) \leqq \bar{\rho}_n(N_f)$. Hence in view of (4.3.40) it follows also that $\bar{\rho}_n(\varphi_n) \leqq \bar{\rho}_n(N_f)$, where $\varphi_n(z) = f(z)e^{-g(z)}$. The converse inequality is obvious. Thus $\bar{\rho}_n(\varphi_n) = \bar{\rho}_n(N_f)$. Q.E.D.

REMARK 1. It is readily seen that *all the assertions of Theorem* 4.3.6 *remain valid for* $\bar{\rho}_n(N_f) = q + 1$, *provided the number* q *is replaced by* $q + 1$ *in the representations* (4.3.34), (4.3.35) *and in the estimate for* $\Psi_n(z)$.

REMARK 2. The integral representations of the functions $\ln \varphi(z)$ and $\ln \varphi_n(z)$ given in Theorems 4.3.4 and 4.3.6 may obviously be regarded as *representations of the entire zero set* χ_f *in terms of its* $(n-1)$*-dimensional submanifolds* $\chi_{f,\tau}$ *and* χ_{f,r^*}.

In conclusion, we remark that the assumptions of Theorems 4.3.5 and 4.3.6 regarding the finiteness of $\rho(N_f)$ and the inequality $\bar{\rho}_n(N_f) < q + 1$ seem superfluous; they are bound up with the specific method adopted here to prove these theorems, and are not intrinsic. However, we know of no proof that would enable us to relax these requirements.

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